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ABSTRACT

This document is divided into two major sections. The first part consists of approximately 60 papers which were presented at the fifth conference of the International Group on the Psychology of Mathematics Education, which was held in Grenoble, France, in July 1981. The papers are categorized into the following major themes: (1) number construction, addition and subtraction, decimals, and geometry in the primary school; (2) proportion and product, algebra, function, and rational numbers in secondary school; (3) problem solution and memory, stages and categories of mathematical thought, logic and representation, and methodological problems; and (4) university teaching, attitude and anxiety, bilingualism, and teacher training. The second part consists primarily of the texts of four plenary lectures which served to outline the main international research orientations. Appended are several papers which missed initial inclusion in the first portion of the document. (MP)

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Groupe international

Proceedings of the
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the International group

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Grenoble, France
13 - 18 juillet 1981

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INTERNATIONAL GROUP FOR THE PSYCHOLOGY OF MATHEMATICS EDUCATION

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PREFACE

The International Group «Psychology of Mathematical Education» (P.M.E.), was formed in 1976 during the 3rd International Congress for the Teaching of Mathematics at Karlsruhe, since then, annual conferences have been held at Utrecht, Osnabrück, Warwick and Berkeley.

The 5th conférence will be held from 13 to 18 july 1981 at the Scientific and Medical University of Grenoble.

The scientific programme of the 5th conference includes :

- five plenary lectures outlining the main international research orientations. The texts of these lectures will be collected in volume II of the congress acts (this will be distributed to participants on arrival at Grenoble).
- approximately 60 papers which will be read in parallel sessions. The texts of these papers are collected in the present volume. They have been classified according to the four following major themes :

A. Number construction, addition and substraction, decimals and geometry, in the primary school.

B. Proportion and product, algebra, function, rational numbers, in secondary education.

C. Problem solution and memory ; stages and categories of mathematical thought ; logic and representation ; methodological problems.

D. University teaching ; attitude and anxiety ; bilingualism ; teacher training.

The order of the papers in the acts will not necessarily be identical with the order in which they are read at the congress. Complete details will be given in the full congress programme.

We would like to thank the following national organisms for their financial backing which has made it possible to offer simultaneous translation in the two official languages, French and English : Centre National de la Recherche Scientifique, Maison des Sciences de l'Homme, Société Mathématique de France, Ministère de l'Education. Thanks are also due to the following local organisations whose generosity will enable participants to appreciate certain cultural and touristic aspects of the Grenoble region : Universite de Grenoble I, Université de Grenoble II, Laboratoire I.M.A.G. et Laboratoire Mathématiques Pures, UER de Psychologie et des Sciences de l'Education, Conseil Général de l'Isère.

Claude COMITI
Gérard VERGNAUD.

PREFACE

Le Groupe International, Psychology of Mathematical Education (P.M.E.), créé en 1976 lors du III^{e} Congrès International sur l'Enseignement des Mathématiques, à Karlsruhe, a tenu depuis lors son colloque annuel à Utrecht, Osnabrück, Warwick et Berkeley.

La 5ème rencontre du groupe se déroulera du 13 au 18 juillet 1981 dans les locaux de l'Université Scientifique et Médicale de Grenoble.

Le programme scientifique du 5ème colloque comporte :

- d'une part, six conférences plénières présentant les principales orientations de recherche sur le plan international. Les textes de ces conférences seront regroupés dans le volume II des actes du colloque (ce volume sera distribué aux congressistes à leur arrivée à Grenoble).
- d'autre part, une soixantaine de communications qui se dérouleront en deux sessions parallèles. Les textes de ces communications sont réunis dans le présent volume, ils ont été regroupés selon les quatre grands thèmes suivants :
  - A. Construction du nombre, addition et soustraction, décimaux et géométrie, à l'école primaire.
  - B. Proportion et produit, algèbre, fonctions, nombres rationnels, dans l'enseignement secondaire.
  - C. Solution de problèmes et mémoire ; étapes et catégories de la pensée mathématique ; logique et représentation ; problèmes de méthode.
  - D. Enseignement supérieur ; attitude et anxiété ; bilinguisme ; formation des enseignants.

L'ordre dans lequel les communications sont imprimées dans les actes ne préjuge pas de celui de leur présentation orale lors du colloque de Grenoble, ce dernier sera précisé dans le programme détaillé du colloque.

Nous tenons à remercier ici les organismes nationaux (Centre National de la Recherche Scientifique, Maison des Sciences de l'Homme, Société Mathématique de France, Ministère de l'Education), grâce au soutien financier desquels nous avons pu organiser pour ce congrès la traduction simultanée de l'ensemble des travaux dans les deux langues officielles (français, anglais) ; nous remercions également les organismes locaux (Université de Grenoble I et Université de Grenoble II, Laboratoires I.M.A.G. et Mathématiques Pures, UER de Psychologie et des Sciences de l'Education, Conseil Général de l'Isère), dont la générosité nous permettra de faire connaître aux congressistes la région grenobloise tant sur le plan touristique que culturel.

Claude COMITI
Gérard VERGNAUD.

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A

CONSTRUCTION DU NOMBRE, ADDITION ET SOUSTRACTION
DECIMAUX ET GEOMETRIE
A L'ECOLE ELEMENTAIRE.

NUMBER CONSTRUCTION, ADDITION AND SUBSTRACTION
DECIMALS AND GEOMETRY
IN THE PRIMARY SCHOOL.

OPERATIONAL COUNTING AND POSITION

Leslie P. Steffe
University of Georgia

Cette communication soutient comme thèse le fait que pour l'enfant, compter est au coeur même de sa conception du nombre. De percevoir leurs façons de compter comme une performance ou même comme l'essentiel des stratégies qu'ils utilisent pour résoudre les problèmes ne permet pas d'entrevoir complètement l'aspect psychologique. Compter est un comportement mathématique significatif et les items que l'enfant considère comme pouvant être compté sont une indication principale de la qualité de cette signification. Les 17 enfants, sujets de cette étude, pouvaient prendre des unités abstraites comme articles dénombrables. Leur conception du nombre se caractérise par une opération "d'unitisation" (intégration) dont le contenu pouvait être des actes dénombrables. Quatre concepts distincts de position permettent de concevoir le dénombrement comme étant plus que performance ou stratégie: (1) la position du dénombrement dans l'ordre croissant; (2) la position d'un dénombrement bi-directionnel; (3) la position d'une déclinaison numérique; (4) la position de la reversibilté de l'extension et de la déclinaison numérique. Ces deux derniers concepts de position sont basés sur des opérations impliquant le dénombrement avant même que ce dernier se produise véritablement. Ce sont ces operations qui lient le dénombrement à la compréhension numérique.

OPERATIONS INVOLVING COUNTING

Counting is linked to numerical understanding by the unitizing operation of integration. An integration is the operation of taking any sequence of counting acts as a whole--as an abstract unit--and is the result of reflective abstraction. A distinction can be made between counting the chimes as a clock strikes the hour and uniting those counting acts into one whole after counting. In the absence of the uniting operation, integration, counting produces at most a sequence. But an integration of a sequence of counting acts yields a specific numerosity.

In the absence of observable counting acts, a child may act in a way that suggests that it re-presents to itself counting acts and integrates them, taking them as a whole. In this case, the child has made a tacit integration. The counting acts are implicit in the sense that the child knows it could carry them out if required. The tacit integration is the act of taking these implicit counting acts as an abstract unit. Finding the sum of eight and seven by counting "eight--nine is 1, ten is 2, ..., fifteen is 7--fifteen" indicates that not only were the first eight counting acts implicit in solution, but that they were treated as one thing--as an abstract unit. That uttering "eight" indicates a tacit integration is made all the more plausible by the double counting behavior. The double counting behavior indicates that the child fully intended to count beyond "eight" seven times before it actually counted. This intention would be possible only if the child took the extension of the eight implicit counting acts as one thing--only if it performed a tacit integration before it double counted. The tacit integration corresponding to "eight" was not expressed. The double counting behavior can be understood as an expression of a tacit integration of the extension of the first eight implicit counting acts. Counting its own counting acts certainly indicates that the child knew what it was doing while doing it. But a child may express a tacit integration without double counting. Moreover, the implicit counting acts may be backward as well as forward.

POSITION

Counting served in different ways as the basis of the concept of the position of an item in a row of items for the 17 children. The analysis of their concept of position was based on their behavior in three tasks.

Task I. The interviewer pointed to the first three of a row of 12 tiles while uttering "1st-2nd-3rd," and then to the ninth tile while uttering "9th." He then pointed to (a) the 10th and asked "Which one is this?" and (b) the 7th and asked "Which one is this?"

Task II. The interviewer covered the first seven tiles in Task I and pointed to the 10th tile and uttered "10th." He then asked (a) "How many are covered?" and (b) "How many are there in all?"

Task III. The interviewer covered the first three of a row of eight discs and then pointed to the fifth disc and uttered "5th." He then asked (a) "How many are there in all?" and (b) "How many are covered?"

POSITION AS COUNTING FORWARD

The most elementary concept of position was based on actual or re-presented forward counting acts. Re-presented counting acts are thought of as the result of re-presenting the actual sensory-motor material out of which the child forms particular counting acts. The physical structure of a counting act--its beginning, its end, and possible intermediary states--is what the child represents to itself (a dynamic representation). A re-presentation of a succession of counting acts may be implied by countable items, extending the child's sensory motor counting activity to implied counting activity.

Two children solved Task I.b by starting at the first tile and counting "1-2-...-7." In Task I.a, they knew that the indicated tile was 10th without having to count all of the tiles, so the visible tiles did suggest forward counting acts to them. These implied counting acts, coupled with dropping-back to "one" and counting forward in Task I.b, certainly indicates the two children were capable of stepping out of their sensory motor stream and re-presenting counting activity.

Both children performed actual counting acts forward over the cover on one occasion in Tasks II or III. One child counted over the cloth "1-2-3-4-5" after counting the five visible discs in Task II.a and then was able to count "10-11-12" in Task II.b. The previously performed forward counting acts "1-2-3-4-5" coupled with the visible tiles preceding and including the tenth were sufficient for her to re-present forward counting acts "1-2-...-9" in Task II.b without actually carrying them out. The other child counted over the cloth and then counted the first two visible tiles, "1-2-3-4-5-6" in Task III.a, where "6" corresponded to the 5th disc. She then made an adjustment in counting and correctly counted "5-6-7-8," where "5" corresponded to the fifth disc. But neither child established a numerosity corresponding to the indicated tile (10th or 5th). The position of a tile could be established only through actual or re-presented forward counting acts.

POSITION AS BIDIRECTIONAL COUNTING

Four children solved Task I.b by starting at the 9th tile and counting "9th-8th-7th--seven," but were limited in the same way as the two children who established position by counting forward when solving Tasks II and III. The four children displayed bidirectional number word sequences, but that alone would be an insufficient explanation of how they solved Task I.b because the two children who established position as counting forward also displayed bidirectional number word sequences. These four children were also capable of bidirectional counting.

Directionality of successive counting acts is established on the sensory-motor level by the passage from a counted item to the next countable item. To form an inverse relation between a particular counting act and its successor, the child must step out of the stream of immediate sensory-motor experience and re-present counting acts. In this case, not only could the child imagine itself perform a counting act, but it could also imagine itself passing from one counting act to the next, in both the forward and the backward direction. It would now be possible for the child, after passing from the counting act "four" to "five," to imagine itself reversing direction and passing back to the imagined counting act "four." This reversibility is similar to Piaget's notion of inversion by reciprocity (Beth & Piaget, 1966, p. 176). If a child landed on the 9th tile in the row after skipping some (Task I), it would realize that to get there, it could have performed a counting act immediately preceding the 9th, i.e. the 8th, etc. In this way, it would know that starting at the 9th and counting to the 1st would produce the same counting acts (which include number word utterances) as starting at the 1st, but in reverse order.

POSITION AS NUMERICAL DECLENSION

Three children used bidirectional counting in Task I.b and also solved one or more of the other tasks. Two solved Task II.a by counting backward "10-9-8-...-1" and one then knew that seven were covered. Both solved Task II.b, the latter by counting "7--8-9-10-11-12" and the former by counting "11-12" (it re-presented the backward counting acts as forward counting acts). All three children solved Task III.a by counting "5-4" and then "4-5-...-8."

The children formed tacit integrations of implicit counting acts backward, which are called *numerical declensions*, and then *expressed* those numerical declensions by counting backward. The numerical declensions did not constitute numerosities because the children had to actually count backward over the cloth to find how many were covered (Task II.a) and even then one did not know how many were covered. Bidirectional counting explains how the children were able to count backward *to the cover* and then turn around and count forward to the end without knowing how many were covered. This behavior does not refute the claim of an initial tacit integration of backward counting acts. The remaining child counted "9-8--8-9-...-12" to solve Task II.a but never knew how many were covered.

POSITION AS REVERSIBILITY OF NUMERICAL EXTENSION AND DECLENSION

Eight children established the position of an item based on reversibility of numerical extension (tacit integration of implicit counting acts forward) and numerical declension. Characteristic solutions of Task III.a and .b were to count "6th-7th-8th--eight," and to count "5th-4th-3rd---three," respectively. The implicit counting acts of the tacit integration corresponding to the discs preceding and including the fifth were constituted as either forward or backward depending on the problem the child thought it was to solve (How many in all? or How many covered?), which is reversibility of numerical extension and numerical declension.

CONCLUSIONS

Operations involving counting were not used by the two children who based position on forward counting. Position of an item in a row of items had to be established by actual counting acts. They did not establish the indicated position of an item (Tasks II and III) by forming an abstract unit of implicit counting acts forward or backward. They were capable of numerical extension as indicated by their behavior in solution to other tasks (double counting), but they never used this basic capacity to establish position. Their meaning of the position of an item in a row was based on the sensory-motor sequence of counting acts forward and consisted of a single counting act of that sequence. These children seemed to never take the sequence of counting acts as a unit. The only advance the four children who based position on bidirectional counting acts had made was that there was no requirement that the sequence of counting acts be actually carried out in toto. But position was still a particular counting act of a sequence, where part or all of that sequence was imagined counting acts. These six children used counting as the substance of a sensory-motor strategy even though they were capable of more.

The three children who established position as a numerical declension realized that they could count backward to the first item even though some preceding items (including the first) were not visible. The implicit counting acts radiated backward to the first item and only needed to be carried out. But their concept of position was not fully elaborated because reversibility was lacking. For the eight children who were reversible, the implicit counting acts not only radiated backward, but also forward depending on the task the child thought it was solving. They operated flexibly and with great power in task solution. Position denoted a numerosity. To characterize their counting activity as substance of a strategy or as a performance would not take into account that, for them, counting was the expression of a tacit integration of implicit counting acts and, as such, was an instrument of number.

In principle, the content of a tacit integration need not be implicit counting acts. A child may unite in thought any collection of unit items. If a child were told that seven robins occupied a nest, not only could the child re-present robins to itself, but it could take them as abstract units and unite these units into a single entity--

a unit composed of units--without counting ever being considered. This distinction is crucial, for counting must be fused with numerical structure to establish specific numerosity.

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THINGS, PLURALITIES, AND COUNTING

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Après une brève exposition d'un modèle théorique, proposé ailleurs, qui explique la formation des concepts relationels et numériques par un mécanisme pulsateur de l'attention, on examine la construction des unités, du singulier, du pluriel, et des collections. Ces structures abstraites, bien qu'elles constituent le squelette des concepts numériques, ne sont que la matière première pour l'établissement des numérosités. D'autre part, le concept de la numérosité, malgré certaines radices préliminaires dans l'expérience sensori-moteur, surgit avec le développement de l'habilité consciente de compter.

Concepts of number, much as concepts of relation, are not determined by the properties of things but rather by what we do in order to experience things in that particular way. To have relations and numbers, however, we must first have discrete things, objects that can be considered unitary. Piaget (1937) has documented how object concepts develop from fuzzy conglomerates in early sensory-motor experience. The process of isolating something from the experiential background has always been taken for granted without analysis. Both Frege (1884) and Husserl (1887) independently expressed the idea that the construction of discrete items constitutes the foundation of the conception of unitary wholes and, ultimately, of countable units; and both Frege and Husserl suggested that it is a conceptual rather than a perceptual act that achieves it.

I have elsewhere presented a model for the construction of units (Glasersfeld, 1981) based on a rhythmic, pulse-like function of attention posited by some neuropsychologists (Craik, 1948; Harter, 1967; Kohlers, 1972). Attention, in that model, does not signify a protracted state but, instead a regular succession of brief moments (at least 7 or 8 per second), each of which can, but need not, register some individual signal. If a signal is registered, the attentional pulse is called 'focused'; if no signal is registered, it is called 'unfocused'. In this model, then, it is a spe-

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cific pattern of focused and unfocused attentional pulses that constitutes unitary items. That pattern can be characterized as beginning with one unfocused pulse followed by at least one pulse focused on some signal, and terminating with one unfocused pulse.

Husserl says that when we speak of the unity of a 'thing', we have in mind a collection of properties that have been taken together to form a 'whole'. Piaget, similarly, insists that when a child forms the concept of an object, this requires the composition of sensory-motor elements from more than one source. Every time, for instance, a child appropriately labels an experience "apple", that experience began with an unfocused pulse, continued with pulses focused on particular visual, tactual, etc., signals and was cut by at least one unfocused pulse from whatever followed in the experiential flow.

Another conceptual step leads from the appropriate use of "apple" to the appropriate use of "apples". In order correctly to use the plural, the child must not only be able to keep track of its 'unitizing' attentional operations, but it must also be able to keep track of particular sensory-motor signals that are being focused on; the unitary things that have been constructed must resemble one another. Provided there are more than one, it is irrelevant how many things there are or whether their number is limited; but it is indispensable that they all conform to a minimal combination of specific signals that has been empirically abstracted as 'apple concept' in prior experience. In other words, the constitution of a plurality involves a process of classification based on a prototype. But a plurality is as yet unbounded and, therefore, has no specific numerosity and must not be mistaken for a 'set'. The prototypic structure specifies the combination of particular signals and determines what can be recognized as a new experiential instantiation of the concept (thus serving as a template for the creation of pluralities).

Let us say, a child recognizes a perceptual situation as satisfying the concept associated with the word "cup". It may utter that word and thus close the experiential episode. But it may also continue by exploring an adjacent part of its visual field and find further similar combinations of perceptual signals. If, in that case, the child has kept track of the fact that its concept of cup was satisfied more than once, it could utter the plural "cups". Conceptually, that plurality would become a collection, if the child perceived the table on which the cups are arrayed as a uniform background that bounds them. At that point, 'the cups on the table' constitute a stable collection that has a certain numerosity -- but that numerosity has not yet been specified. In order to establish its numerosity, the collection would have to be counted. To count a collection is more than to perceive it. It requires the coordination of number words to the unitary things that are to be counted. Neither the production of the standard number-word sequence (cf. Fuson & Richards, 1980) nor the various types of of unitary items that may be counted (cf. Steffe et al., 1981a) will be discussed here. Rather, I shall try to explicate the abstracting activity that turns a collection of sensory-motor things into arithmetic units.

First, sensory-motor things, e.g. cups, must be considered qua unitary items, that is, insofar as they were constituted by the pattern of attentional pulses that cut them out of the child's experiential field. Steffe et al. (1981b) provided an elegant experimental confirmation of the assumption that the recognition of perceptual things as acceptable candidates in the formation of a collection is preliminary to the act of counting them as units. This second step is an operation of abstraction, in that it reviews things that have already been constituted and abstracts from them the one feature that is crucial for the purpose of counting, namely the attentional pattern that shaped the sensory-motor signals into discrete unitary items. The result of that abstraction, the attentional pattern as such, I have called a lot, i.e. a compound conceptual structure of unitary items that can be counted by the simple iterative alternation of focused and unfocused attentional pulses.

The last step in the construction of a number concepts involves a further operation of abstraction that takes an iterative attentional pattern (a lot) and, in reviewing it, superimposes on it the attentional pattern that constitutes unitary wholes. This operation is of the kind that Piaget called reflective abstraction (1970, p.18) and it produces a structure that is a unit and, at the same time, is itself composed of units.

So far I have briefly outlined the hypothesized conceptual structures which, I believe, satisfy all requirements of the concepts we call "number". I want to stress that the attentional model and the structures it produces do not account for the concept of numerosity or the quantitative aspect of collections, lots, or numbers. In the perspective of the Georgia Project, that quantitative aspect is the result of a developmental progression in the counting activities of the child. On the other hand, there are experiential roots of the concept of quantity that precede the use of numbers and number words. In this second part of the paper I shall survey several phenomena that belong to an area that could be called "elementary experience of numerosity".

Recent experiments by Starkey & Cooper (1980), suggesting that infants of about six months reliably discriminate between linear arrays of two and three dots, irrespective of the length of the arrays, have startled some researchers in mathematics education. In our view, that finding is not so surprising. At least half a dozen studies show that monkeys and apes do as well, if not better, in discriminating arrays of two or three items (Dooley & Gill, 1977; Ferster, 1964; Hayes & Nissen, 1971; Thomas et al., 1980). It seems clear that this ability is not contingent upon any kind of numerical system, let alone number words or counting. Discriminations of that kind could be made by differentially tuned neurons: some that fire when they receive two successive impulses, others that fire only when they receive thre Such simple computational devices are a commonplace assumption in neurophysiology. In addition there is the well-known human (and animal) ability to recognize and accurately recall rhythms of one, two, and three beats in music, dance and poetry. Hence it seems a reasonable hypothesis to assume tha the nervous system has the built-in capability of distinguishing between sequences of one, two, or three signals in any sensory mode, including the kinesthetic. If that is the case, however, such discrimination must not be taken as evidence of numerical concepts, even if the subjects have associa ed number words with the respective events. The reason for this is simply that, given such a built-in computational ability, the correct recognition discrimination, and naming of these events does not require the knowledge that, say, the event called "three" is a unit comprising a plurality of units. In this respect the recognition of rhythmic configurations is analogous to the phenomenon of 'subitizing', which concerns the recognition of spatial configurations that have been associated with number words.

The association of spatial patterns and number words can arise in many ways. Dominoes, playing cards, and other games involve the recognition of conventional configurations of dots and other unitary elements, and in man instances these configurations have names that are number words. Children, thus, may learn that the name of a particular pattern on playing cards is "five", and they can learn this in exactly the same way in which they lear that the name of the written numeral 5 is "five". That is to say, that semantic link can be acquired *without* any awareness of the fact that the pattern on the playing card is a composite of five unitary elements. At some later point, however, the child will *count* the elements -- and when that happens it will give rise to the kind of discovery that functions as one of the mainsprings of cognitive development: the discovery that two hitherto separate schemes -- in this case, naming the card and counting the elements on it -- lead to one and the same result, namely the word "five". The material occasion of the discovery is irrelevant. In whatever circumstances it occurs, it will provide the first and most immediate revelation of the conceptual fact that the number word refers both to a unitary thing (the card) and to a collection of units (hearts, spades, etc.).

In one special, limited sense that experience may occur much earlier, on the sensory-motor level, without number words and therefore without revealing anything about the structure of numbers. George Forman, working with very young children of eight to twelve months, has minutely documented and analysed the various things they do with building blocks (Forman, 1973; Forman et al., 1975). Through his observations it has become clear that there is a moment when children discover that a transposition of two blocks can be achieved in two different ways: by two sequential movements of one hand and by the simultaneous movement of both hands. Since certain simultaneous movements of hands are often experienced as *one* movement (owing to the bilateral symmetry of the motor system), there are situations where "oneness" and "twoness" fall together in a single experience which, at a later stage, may give rise to reflection and thus to immediate experience of that characteristic structure of number.

Such immediate experience of the dual (unitary versus compositional) structure of a result, could be called"proto-numerical". It is developmentally parallel with, but conceptually different from, the distinction that Piaget has analysed under the heading of "intensive" and "extensive" quantity (Piaget & Szeminska, 1941). Under all circumstances, however, this proto-numerical knowledge is limited to small numbers, in most cases below four or five, and its conceptual realization, or *prise de conscience*, is brought about by an act of counting. Hence it is counting that functions as the main instrument in the conceptualization of numerosity.

In his theory of counting types, Steffe (e.g. Steffe et al., 1979) has developed an anatomy of *what* is being counted by the child. He has posited a progression from perceptual items to abstract units, and in that progression, the concept of number originates at the point when the child becomes aware of the fact that any number word *implies* the possibility of arriving at that same number word by way of a sequence of counting acts determined by the standard number-word sequence that precedes it. Counting acts, in that theory, involve the coordination of number words and unitary items of *some* kind, i.e., items that are unitary because their production, be it perceptual, kinesthetic, verbal, or through a re-presentation of any of these, is the result of the application of the attentional pattern that creates discrete units. Given this coordination, the child's knowledge that, for instance, saying "seven", implies the possibility of arriving at "seven" by uttering the standard number-word sequence up to seven, comes to be complemented by the knowledge that, with each number word of that sequence, an individual unitary item could be coordinated. The number word sequence preceding any given number word, therefore, comes to represent a potential collection of things and, ultimately, a *number* of abstract units. Only when the knowledge of these inherent implications has been fully grasped and is available at any time, can the child be said to possess concepts of number and of numerosity.

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PRE-NUMERICAL COUNTING

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Le dénombrement: la production d'une séquence de mots-nombres telle que chaque mot-nombre s'accompagne de la production d'un élément-unité (unit item). Le dénombrement se distingue du comptage (rote-counting), c.a.d., la récitation en isolement d'une séquence de mots-nombres. Ce travail fournit une analyze conceptuelle de l'acte de dénombrement chez l'enfant pré-numérique. Pour compter, l'enfant doit produire une catégorie d'éléments dénombrables. L'élément dénombrable se co-ordonne avec un mot-nombre, et se transforme en élément dénombré. Cet accomplissement considérable précède de longtemps, néanmoins, la construction du nombre.

> Counting is the production of a number word such that each number word is accompanied by the production of a unit item.

Counting, thus conceived, is a complex activity which can be both more and less than meets the eye. On the one hand, a child may "rote-count" (cf. Steffe, Richards and von Glasersfeld, 1980), that is, the child may simply recite a number word sequence. From the perspective of the above definition, while this may appear to be an advanced sort of behavior, it remains in the realm of language. Reciting a number word sequence is not considered "counting" because the number words are not accompanied by the production of unit items. On the other hand, a child may engage in quite complex counting behavior, coordinating the number word sequence with a wide variety of actions and yet be unable to use counting as an instrument of number.

In this paper I focus on the nature of counting for children who are pre-numerical. Counting for these children is an activity which becomes progressively more complex and sophisticated -- ranging from rote-counting to counting objects of all sorts, to counting representatives of objects. This is a tremendous achievement for a child who is, by and large, still unable to describe or explain what he is doing or how he is counting. This coincides with Gelman's insight that children know more than they can tell, and that, as a result, "...the cognitive capacities of preschoolers have been underestimated" (Gelman and Gallistel, 1978, p. 13). I argue here that the use of counting, for a child who is pre-numerical, is inherently limited by the childs inability to construct number, *per se*. Nevertheless, there is a distinct character to the nature and functions of counting for these children which provide some clues regarding the childs understanding of their own actions.

ROTE-COUNTING. Rote-counting consists in the recitation of number words in sequence. The words in the sequence are not accompanied by, or coordinated with, events or actions in another sequence. Number word sequences are produced as early as age two. The child, at this point, does not match this sequence with other activities in order to count anything. Thus, even though the child may rote-count while hopping, or skipping, or clapping hands, the number words are nothing more than ordered nonsense syllables. The important distinction between rote-counting and counting, in our terms, is that there is no production of unit items. The number word sequence is isolated and is recited as a poem, or as a song.

The construction of the number word sequence is complex, and is, in its own right, a subject of investigation (cf. Fuson and Richards, 1980; Steffe, von Glasersfeld and Richards, 1981). However, the point to be made here is that the recitation of the number word sequence by itself is to be carefully distinguished from counting. For the child who produces isolated number word sequences, the words in the sequence have a "sequence meaning". They are not standing for or referring to numbers, or operations on numbers, or even to a part of counting activity (recent work reported by H. Sinclair suggests that in the beginning counter the number words are taken as names of the objects to which they are coordinated). After acquiring an initial segment of the number word sequence (being able to repeat it in conventional order), the learned part of the sequence is elaborated by the child through the construction of relations such as "comes after", "comes right after", "comes before", and "comes between". Clearly there is a numerical interpretation of these relations (e.g. "successor"), but it would be a mistake for an observing adult to assume that this is the child's understanding. The sequence relations are precursors of the numerical relations, and serve as a basis for the child to learn the numerical relations, but it would be a mistake to ignore the differences.

COUNTING. The earliest form of counting occurs when the child takes perceptual items in its experience as units to be counted. A child who is only able to count items in their perceptual field we call a "counter with perceptual unit items". Counting of any sort, though, still requires that the child establish, prior to counting, a category of countable items -- of what it is that can be counted. Counting, as a result, is conceptual even for these children.

In order to begin counting, the child must have a concept of what is to be counted. This concept -- what we have called a "template" -- is the result of an abstraction from experience. In Piaget's terms it is an "empirical abstraction". When a particular item is taken as falling under the concept then it may be counted. Each countable item must be the same with respect to the concept, that is, the item must be taken, by the child, as one more instance of the concept. The child must take the item as a unit, as a single thing, no matter how complex the item might appear given a different context. Taking an item as a unit is an action of the child. The item must be considered as a unit to which, in the act of counting, a single number word will be coordinated. In spite of the conceptual nature of counting, even at this primitive level, there is an important rote aspect to the behavior.

Counting, for a counter with perceptual unit items, works very much like a script (cf. Schank and Abelson, 1977). The child is learning a role to play in certain situations. Counting is an appropriate response to questions like "How many?", or "Which?", or "Can you count these?". This is by no means a simple role, for it involves the coordination of very different motor sequences. As the child becomes better at playing the part, other nuances become important: emphasize the last number word uttered; remember the last word uttered; do not count something twice (and in general be careful about the correspondence between the unit item and the number word); indicating (nodding or pointing) is as good a tag as grasping; and so on. Eventually the child comes to be able to re-present items which are not perceived, and to count the representatives. The child will also be able to carry the counting script further, e.g. to combine two such scripts. Initially "How many?" may draw a response of "1,2,3,4,5,6" (while pointing to some objects in synchrony with the utterances). Later the child learns that the more appropriate response is "1,2,3,4,5,6 ...6", or perhaps, better yet, "6", and further, if the objects are hidden, it is possible to extend fingers sequentially while uttering the number words. Here the fingers function as representatives of the hidden objects.

In these activities, the child is learning how to use counting, and the child is learning how to react to (what from the adult's perspective are) different numerical situations. It is inappropriate to use adult concepts like cardinality to describe what the child is doing. The child no more understands a "cardinal principle" because he says the last word of a counting sequence loudly, than he understands an "ordinal principle" because he says the number words in order. While the ability to answer "How many?" ("Which?") type questions is clearly a part of what will become cardinality (ordinality) for the pre-numerical child, it is perhaps more accurate to refer to this as "pre-cardinal" ("pre-ordinal") understanding.

NUMBER. Euclid observed that a number is a unit (a multitude) which itself is composed of units. This observation is the basis for our own theoretical understanding of the construction of numerical concepts in the child. A particular number is produced as a double act of abstraction (cf. Steffe, von Glasersfeld and Richards, 1981, ch.1). Number so conceived is a process -- the process of taking a multitude of units as a unit. We emphasize the process aspect of this by speaking of "unitizing", or by speaking of "taking an item as a unit." Both of these expressions bring out the activity of forming a unit -- this is a conceptual creation on the part of the child. When a child is able to make this double abstraction there is a marked change in the child's mathematical reality. The child has reflfected on his own actions in counting, and abstracted what is common in them -- the construction of a unit item. Through this reflection the child is made self-conscious of its actions in counting, and is able to provide better descriptions and explanations of its actions. This also results in a greater flexibility in chosing which way to count, and in chosing different strategies for using counting. All of this shows up directly in the problem solving strategies the child employs (cf. Steffe, Thompson, and Richards, 1981; and Steffe, von Glasersfeld, and Richards, 1981).

The second level of abstraction, reflective abstraction (in the sense of Piaget), frees the child from a reliance on specific sensory-motor experience. The child is able to use any sensory-motor material as a basis for counting. Moreover, the child is able to unite, or integrate, the separate actions in counting into a whole. Counting to seven, or merely uttering "seven", produces the same result, viz., a unit which has distinct components (in this case seven of them). This provides the conceptual basis for counting-on. That is, if the child were to add 7 and 5 it would not be necessary to count first to 7, and then five more. Uttering "seven" would be sufficient for the construction of the first addend. The child can then continue counting, and in the process keep track of the second addend. In this case "seven" replaces the counting actions which could be carried out if the child felt the need. It is important to stress that counting-on, as a script, can be taught to children who are not numerical. The child who understands counting-on, however, must be numerical. While this is clearly difficult to assess in practice, it is possible to alter the context of the problems, so that the child will not see the problems as the same. (For example, we distinguish between "How much is 7 and 5?" and a problem where there are seven checkers hidden under one cloth, and five under another. The former problem may serve to key a "count-on script", which may be avoided by the non-standard nature of the other problem.)

FINAL THOUGHTS. Children's counting behavior is extremely complex, and, certainly, counting is a major achievement of early childhood. Researchers have only recently actually looked at the complexity and variety of counting in children. This is due to several features of children's counting which make the child appear to know both more and less than he understands.

First, observable behaviors in counting are remarkably similar, even between a child of three and an adult. Both appear to be doing the same thing. But there are important subtle differences in the behavior which reflects what -- from the counter's perspective -- is being counted. And differences in what is being counted reflect radically different cognitive structures underlying the behavior. While it may seem obvious that there are fundamental cognitive differences between an adult and a child, it has not been accepted that these differences are reflected in counting.

Second, our language to describe counting, and other numerical-like behavior is tied to adult conceptions, rather than childrens'. As a result there is a tendency to assume that there is adult type understanding on the part of the child.

Finally, the child's understanding of counting appears to precede her ability to describe this understanding. Description requires that the child be able to reflect on her own actions and this itself appears as a rather late development in learning the counting process. When the child is able to reflect on her own counting, she is close to actually constructing number, and has been successful in counting for quite some time.

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ETUDE DU FONCTIONNEMENT DE CERTAINES PROPRIETES DE LA SUITE DES NOMBRES DANS LE DOMAINE NUMERIQUE [1,30] CHEZ DES ELEVES DE FIN DE PREMIERE ANNEE DE L'ECOLE OBLIGATOIRE EN FRANCE. (COURS PREPARATOIRE)

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Abstract : The aim of this research is the clarification of the conditions whereby the learner, in the first year compulsory education (cours préparatoire) constructs and internalizes the concept of natural number. The research is based on a close and rigorous analysis oflearner's behaviour when confronted with problem situations designed according to "a priori" established priorities. This analysis is based on one-to-one interviews, method, which enables the dynamics of the processes involved for each learner, and thus the evolution of his cognitive system with respect to the task in hand, to be apprehended.

In this paper, in a problem situation called, "The Race", we examine the functioning of ordinal properties of natural numbers, the acquisition of which plays an important role, both at the level of the construction of the concept of number and at the operational level. We show, in particular, that while the majority of learners at the end of the "Cours Préparatoire" are capable of supplying the preceding and following numbers in the numerical field [1,30], and that while they can count starting form a number other than one within the field, they nevertheless experience numerous difficulties with respect to the functioning of properties, such as, for example, cardinal-ordinal relations, or the property of rank invariance.

Furthermore, even when one of the properties under consideration functions in a given situation, this, in itself, does not mean that the learner has acquired it. The learner may, at a later date, no longer be able to use it or, alternatively, when the situation is slightly modified he may overgeneralize as the functioning of these properties depends both on the learner's temporary mathematical competence and on the tasks which confront him.

Depuis un certain nombre d'années, notre équipe conduit une recherche dont le but est *la clarification des conditions dans lesquelles l'élève du Cours Préparatoire* (première année de l'enseignement obligatoire en France, C.P.) *construit et s'approprie le concept de naturel*. Nos principaux objectifs sont :

- mettre en évidence les différents modèles implicites et incomplets fonctionnant chez les élèves à un moment donné dans une situation mettant en jeu ce nombre ;
- comprendre comment les différents modèles qui coexistent, à un moment donné, chez un même enfant, fonctionnent selon la tâche à résoudre.

Nous appelons ici modèle implicite l'ensemble des invariants, des propriétés du concept de nombre, des relations, dont on peut *inférer* le fonctionnement chez l'enfant *à partir des actions de ce dernier* dans des situations où intervient ce concept.
La recherche de tels modèles suppose l'hypothèse que, quelles que soient les conditions dans lesquelles sont faits les apprentissages scolaires, il existe, à un certain niveau, *des régularités* qui caractérisent l'appropriation d'une connaissance donnée chez tous les sujets.
Les différents modèles qui coexistent, à un moment donné, chez un même enfant, constituent *son système de connaissance provisoire*. Mais ce système de connaissances ne contient pas en lui-même toutes les conditions de l'utilisation de ces connaissances. C'est pourquoi il est indispensable d'étudier *les différents fonctionnements de ce système selon la tâche* à laquelle l'élève est confronté.

La formulation des modèles implicites du concept de nombre est assujettie à diverses analyses interdépendantes :

. une analyse mathématique du concept de naturel dans le but d'en caractériser les propriétés et les invariants afin de permettre un choix des tâches auxquelles on confrontera l'élève.
. une analyse des tâches retenues afin de dégager des classes de procédures possibles relativement à chacune de celles-ci.
. une analyse du système d'interaction "élève-tâche" afin de mettre en évidence certaines variables de la situation pouvant modifier le rapport du sujet à la tâche.
. une analyse des comportements du sujet dans une tâche donnée, en termes de procédures et de types d'erreurs, analyse qui doit permettre de vérifier ou d'infirmer les hypothèses faites a priori sur le fonctionnement de système de connaissance de l'enfant face à la situation construite précédemment.

L'objet de notre communication sera d'essayer de montrer comment nous avons mis en oeuvre la problématique brièvement résumée ci-dessus pour étudier le fonctionnement de la suite des nombres et de ses propriétés chez l'élève de fin de C.P. confronté à une situation-problème : "La course". La population étudiée comprend 58 enfants issus de trois classes de C.P. de Grenoble et de sa banlieue.

Avant de placer les enfants face à la situation-problème en question, il nous était indispensable de connaître le domaine numérique dans lequel ces derniers maîtrisaient la récitation de la suite des nombres (à partir du début de celle-ci). C'est pourquoi l'entretien débutait par les questions "Tu sais compter ?" - "Jusqu'où ?" - "Montre-moi". Ce sont les réponses des enfants à ces questions qui nous ont permis de répartir notre population en trois catégories, les catégories I, II et III, qui nous serviront de référence pour suivre le comportement des élèves lors de l'épreuve "La course". Se trouvent donc en :

Catégorie I, 28 enfants ne faisant aucune erreur, sinon accidentelle, jusqu'à soixante

Catégorie II, 12 enfants ne récitant la suite des nombres sans errreur que jusqu'à trente, mais soit s'arrêtant là, soit multipliant des erreurs entre trente et soixante.

Catégorie III, 18 enfants incapables de réciter la suite des nombres sans erreur jusqu'à trente.

1 - Analyse mathématique des propriétés dont nous voulions étudier le fonctionnement chez l'élève

Propriété 1 : dans une file (ensemble fini d'éléments matériellement rangés les uns à la suite des autres), le rang du dernier élément est égal au nombre d'éléments de la file.

Propriété 2 : dans une file, tout élément, hormis le dernier, a un suivant ; le rang du suivant de l'élément de rang p est p+1.

Propriété 3 : dans une file, tout élément, hormis le premier, a un précédent ; le rang du précédent de l'élément de rang p est p-1.

Propriété 4 : dans une file, lorsque l'on connaît le rang de l'élément x, on peut déterminer le rang de tout élément placé dans la file après x en comptant à partir du rang de x.

Propriété 5 ou lien cardinal-ordinal : étant donné une file F et un ensemble E de même cardinal n, quelle que soit la bijection de E sur F, f, si d est le dernier élément de F, l'ordinal de l'élément de E qui a pour image par f, d, i.e. $f^{-1}(d)$ est égal à n.

Propriété 6 ou propriété d'invariance : étant donnés une file F, un ensemble E de même cardinal et deux bijections f et g de E sur F, si x est un élément de F, les deux éléments de E $f^{-1}(x)$ et $g^{-1}(x)$ ont le même ordinal : c'est le rang de x dans F. En particulier si x est le dernier élément de F, d, $f^{-1}(d)$ et $g^{-1}(d)$ ont même ordinal.

2 - Description de la situation problème "La course"

Dans cette épreuve, l'ensemble E en jeu est un ensemble de trente cartes rectangulaires sur lesquelles sont écrits trente prénoms différents. Chaque carte représente donc un coureur.

Une bande orientée de gauche à droite, sur laquelle sont dessinées trente cases (chaque case a le même format rectangulaire que les cartes) sert de file de référence F ; la 18e case est coloriée en vert. Cette file permettra de matérialiser l'ordre d'arrivée des coureurs à chaque étape de la course : il suffira pour cela de déposer sur les cases les cartes portant les noms des coureurs selon leur ordre d'arrivée à l'étape.

Dans la 1ère étape, nous étudions la mise en oeuvre des propriétés 1, 2, 3, 4 et 5, lors de la numérotation du "maillot vert" (coureur arrivant sur la case verte) et du dernier, ainsi que des précédents et suivants de ces deux coureurs. La 2ème étape correspond à une deuxième bijection, différente de la précédente, de E sur F ; nous allons là étudier la mise en oeuvre des propriétés 5 et 6, lors de la numérotation du maillot vert et du dernier à cette étape. La 3ème étape n'est courue que par 29 coureurs, l'un d'entre eux ayant abandonné la course. La difficulté de cette dernière étape réside donc dans le fait que l'élève doit structurer le nouvel ensemble de coureurs, non pas par une bijection entre cet ensemble et la file F, mais entre cet ensemble et une nouvelle file F' constituée par les 29 premières cases de F. Le questionnement porte là encore sur le rang du dernier et celui du maillot vert. L'analyse détaillée du fonctionnement des propriétés 1, 2, 3, 4, 5 et 6 a donné lieu à un article à paraître prochainement, *nous nous bornerons dans ce qui suit à l'étude du fonctionnement des propriétés* 5 *et* 6.

3 - Fonctionnement des propriétés 5 et 6 lors des deux premières étapes

Où interviennent ces propriétés ? la propriété 5 peut intervenir dès la 1ère étape, puis dans la 2ème étape, lors de la numérotation du dernier, quant à la propriété 6, elle ne peut intervenir que dans la 2ème étape, pour la numérotation du maillot vert et du dernier, puisque c'est une propriété d'invariance du rang.

Etude de la numérotation du dernier dans les deux premières étapes

Pour obtenir le numéro du dernier *dans la 1ère étape*, deux méthodes peuvent être utilisées :

- utiliser le lien cardinal ordinal (propriété 5) puisque les enfants ont précédemment dénombré le nombre de coureurs ;
- compter à partir de 1 ou d'un nombre intermédiaire, par exemple à partir du numéro 19 précédemment attribué au maillot vert. Cette méthode était déconseillée par l'expérimentation qui disait "Tu as vu, Z est dernier, peux-tu lui mettre son numéro *sans compter ?*"

Résultats obtenus à la 1ère étape

catégorie comportement	I et II	III	Σ
propriété 5	21	8	29
comptage	19	6	25
rien	0	4	4
Σ	40	18	58

Il nous paraît intéressant de noter ici que les 29 enfants ayant mis en oeuvre le lien cardinal-ordinal (soit la moitié de la population) appartiennent en même proportion à la catégorie III qu'aux catégories I et II.

Dans la 2ème étape, l'expérimentateur se contente de dire à l'enfant "A est dernier, mets lui son numéro". Pour ce faire, l'enfant peut, soit compter, soit répondre spontanément 30 en utilisant soit le lien cardinal-ordinal, soit l'invariance du rang du dernier, soit encore la coordination de ces deux propriétés.

Résultats obtenus à la 2ème étape

comportement / catégorie	I et II	III	Σ
réponse spontanée	36	10	46
comptage	1	2	3
rien	3	6	9
Σ	40	18	58

Contrairement à ce qui se passait précédemment, très peu d'enfants comptent alors que cette fois, le comptage n'était pas déconseillé. Pour pouvoir mieux interpréter le comportement des 46 réponses spontanées, étudions l'argumentation fournie en réponse à la question de l'observateur "Pourquoi as-tu mis ce numéro ?".

Ces argumentations, lorsqu'elles ne sont pas ambiguës, ce qui est le cas de 5 d'entre elles, se répartissent comme suit :

- argumentation s'appuyant sur la propriété 5 : "On n'en a pas enlevé, on n'en a pas ajouté, il y en a toujours trente, alors le dernier, il a trente" ; 30 enfants utilisent ce type d'argumentation.
- argumentation s'appuyant sur la propriété 6 : "tout à l'heure, l'autre dernier, il avait le numéro trente" ; 11 enfants utilisent de telles argumentations.

Nous n'étudierons pas ici la numérotation du maillot vert mais nous pouvons donner les résultats globaux suivants obtenus sur l'ensemble des deux premières épreuves : sur les 58 enfants :

- 22 ont mis en oeuvre la propriété 5 et 6 aux moments pertinents
- 20 ont mis en oeuvre la propriété 5 sans jamais mettre en oeuvre la propriété 6
- 8 ont mis en oeuvre la propriété 6 sans jamais faire fonctionner la propriété 5
- 8 (tous de catégorie III à l'exception de l'un d'entre eux) n'ont jamais fait fonctionner aucune des propriétés en question.

4 - Etude de la 3ème étape de la course

Description et analyse a priori de la 3ème étape

Nous expliquions aux enfants que l'un des coureurs épuisé, Jean, abandonnait la course à la fin de la 2ème étape. L'enfant retirait donc du paquet de cartons celui de Jean et le déposait bien en vue sur la table. Les autres cartons étaient réunis en paquet. Puis, nous disions en montrant un carton sur lequel était par exemple écrit Louis : "à la 3ème étape, c'est Louis qui arrive le dernier, écris son numéro".

Notre objectif était d'examiner, dans cette nouvelle situation (un coureur de moins) le fonctionnement de la propriété 5, i.e. du lien cardinal-ordinal. Pour provoquer ce fonctionnement, nous avons essayé de bloquer toute procédure de comptage : pour cela, contrairement aux deux étapes précédentes, l'élève ne déposait plus les cartons des coureurs sur les cases de la bande selon l'ordre d'arrivée : cet ordre d'arrivée n'était donc plus matérialisé.

Examinons a priori l'ensemble des propositions conduisant à la réponse correcte :

- il y a changement de situation par rapport aux deux étapes précédentes (un coureur de moins) ce qui entraîne une modification du cardinal de l'ensemble des coureurs ;
- il y a un coureur de moins, donc le cardinal cherché est le précédent de 30;
- le précédent de 30 est 29 (remarquons que ceci exige non seulement le fonctionnement de la propriété 3 mais un fonctionnement *correct* de cette propriété, ce que nous noterons à partir d'ici 3^c).
- le rang du dernier est égal au cardinal du nouvel ensemble (propriété 5)

Hypothèses

Les propositions énoncées ci-dessus nous permettent de prévoir que le fonctionnement des propriétés 3^c et 5 va jouer un rôle déterminant à cette étape : nous caractériserons donc le comportement des enfants à partir du fonctionnement de ces propriétés lors des deux premières étapes ; ceci partage notre population en deux groupes :

- 27 enfants de comportement que nous noterons "type 1" : (3^c, 7)
- 25 enfants de comportement que nous noterons "type 2" : (3^c, non 7) ou (non 3^c, 7) ou (non 3^c, non 7).

Une condition nécessaire au succès est que les enfants aient bien perçu le changement de situation. Lorsque cette condition est remplie, nous faisons l'hypothèse que seuls les comportements de type 1 peuvent conduire au succès spontané (c'est-à-dire à la réponse 29).

Résultats obtenus

Réponse spontanée type	29	30	autre	Σ
1	11	16	0	27
2	1	21	3	25
Σ	12	37	3	52

* Notons tout d'abord que seulement 12 enfants répondent spontanément 29. Tous les succès, sauf 1, concernent des enfants de type 1. Quant à l'exception DEL, son étude montre que sa très bonne maîtrise du comptage l'a toujours amenée à utiliser ce dernier (de la manière la plus économique, c'est-à-dire en utilisant la propriété 4) pour trouver les précédents demandés dans les étapes précédentes. Ici, les possibilités de comptage

étant bloquées, elle trouve immédiatement le précédent de 30, montrant que la non mise en oeuvre précédemment de la propriété 3 ne signifiait pas la non acquisition de cette dernière.

- Il reste que 37 enfants, dont 16 de type 1 répondent 30. Une étude plus poussée, que nous nous réservons d'exposer oralement, montre que certains enfants n'ont pas vraiment réalisé qu'il y a à cette étape un coureur de moins alors que pour d'autres, le rang du dernier semble être définitivement rattaché au rang de la dernière case de la bande et ne pas défendre du cardinal de l'ensemble.

  Le questionnement de l'expérimentateur "Combien y a t-il de coureurs à cette étape ?" amène un grand nombre d'enfants à fournir finalement la réponse exacte. Il reste que 7 enfants n'établissent malgré cette question aucun lien entre le cardinal du nouvel ensemble et le rang du dernier, puisqu'ils continuent à lui attribuer le numéro 30 ! Ce comportement peut d'autant plus paraître surprenant que certains de ces enfants avaient fait fonctionner le lien cardinal-ordinal dans les étapes précédentes. Cela prouve que pour ces enfants, il n'y a pas véritable appropriation de la propriété 5, puisque le fonctionnement de cette dernière dépend des variables de la situation (bijection matérialisée ou non).

Il ne faudrait donc pas croire que certaines propriétés de la suite des nombres sont définitivement acquises parce que les enfants les ont mis en oeuvre dans une situation donnée, une simple modification de cette situation pouvant suffire à bloquer le fonctionnement de ces propriétés.

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L'ENFANT ET LE COMPTAGE

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Abstract : This paper is extracted from a study (Fischer, to be published) more extended about the child's counting. We only give here some information the experimental method and some results about this study, which concerned 224 children, age 3 years to 6 years 6 months, frequenting "école maternelle" (french infant school).

These results concern on the one hand the development of child's counting (reciting of the sequence of whole numbers, application of the "how-to-count" principles defined by Gelman and Gallistel), and on the other hand the function ofthe counting in the denomination of the first numbers.

For the first of these points, we present a table which describes the application of the "how-to-count principles" as a function of the numbers' size and of the children's age. We also give a graphical representation of the application rate of the set of the "how-to-count principles", for a size like five, as a fonction of the children's age.

For the second point we present an other table, always as a function of the children's age and of the numbers' size, giving the children's behaviour (exteriorized counting or not), and 3 tables, which allowed us to prove that a child who knows how to name (as an answer to the question : How many counters ?) a number n (= 3, 5 or 7) also, almost surely, how to count a set of n counters.

Our conclusion, which is also the result of the complete study, is that Instruction should more take into account a phenomenon as important as the child's counting.

Introduction : Dans une étude (Fischer, à paraître) beaucoup plus vaste, nous avons :

- décrit le développement du comptage chez l'enfant, en particulier les principes du comptage dégagés par Gelman et Gallistel (Gelman, 1978);
- émis l'hypothèse d'un rôle important du comptage dans la dénomination des premiers nombres et dans la résolution des premiers problèmes par l'enfant;
- souligné quelques caractéristiques, fonctions ou conséquences du comptage;
- analysé comment la didactique des mathématiques avait tenu et tient compte de ce phénomène important qu'est le comptage de l'enfant.

Dans la présente communication nous nous contenterons de donner quelques précisions expérimentales et quelques résultats (concernant les 2 premiers points mentionnés ci-dessus) relatifs à cette étude.

I PRINCIPES DU COMPTAGE ET DESCRIPTION DE L'EXPERIENCE

1) Principes du comptage : Nous reprenons ceux dégagés par Gelman et Gallistel (Gelman, 1978). Précisons simplement nos traductions et notations :

the one-one principle → le principe de bijection (pour n), noté $P_{bi}(n)$
the stable-order principle → le principe de suite stable, $P_{ss}(n)$
the cardinal principle → le principe cardinal, $P_{ca}(n)$
The how-to-count principles → Les principes du "Comment compter".

2) Description de l'expérience

a) La population : Le même expérimentateur a interrogé individuellement 224 enfants (milieux socio-culturels variés) fréquentant l'école maternelle, d'une assez grande ville et de sa banlieue. Ces enfants sont répartis en 7 Groupes d'Age (G.A), chacun de ces G.A comprenant 32 enfants et couvrant 6 mois. Nous noterons G.A 6;3 le G.A comprenant les 32 enfants entre 6 ans et 6 ans 6 mois, et appellerons (notations analogues pour les autres G.A) grands les 96 enfants des G.A 6;3, 5;9, 5;3, moyens les 64 enfants des G.A 4;9, 4;3, et petits les 64 enfants des G.A 3;9, 3;3.

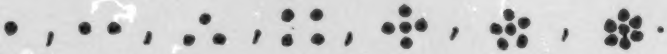
b) Les épreuves : Chaque enfant passe les 4 épreuves (dans l'ordre) suivantes :

Epreuve 1 : Récitation de la suite des nombres

On demande à l'enfant s'il sait compter, jusqu'à combien, et, le cas échéant, de le montrer. S'il ne récite pas, ou mal (sans commencer par 1, 2), on amorce la récitation : "1, 2, comme ça". S'il s'arrête sans s'être trompé, on l'incite à continuer : "Et après ?". On arrête la récitation systématique après 30.

Epreuve 2 : Dénomination des nombres

On présente à l'enfant des jetons en bois dans une boîte circulaire et dans les configurations approximatives suivantes :



Les nombres de jetons et l'ordre de présentation sont les suivants : pour les grands 7, 5, 3, 6 et 4; pour les moyens 5, 3, 4 et 2; pour les petits 3, 1, 4 et 2. On demande à l'enfant combien il y a de jetons dans la boîte sans lui suggérer de compter. Si l'enfant se trompe, on lui demande s'il est sûr et on l'encourage, si on a l'impression que l'échec est accidentel, à reconsidérer sa réponse.

Epreuve 3 : Résolution de problèmes (non présentée dans cet article)

Epreuve 4 : Comptage-pointage induit

L'expérimentateur présente à nouveau à l'enfant la boîte avec les jetons et lui demande : "Tu sais compter comme ça ?" en pointant 2 des jetons avec l'index. Si alors l'enfant ne compte-pointe pas, ou mal, l'expérimentateur recommence un comptage-pointage accompagné de "1, 2" (sans dépasser 2). En dernier recours, il guide même le doigt de l'enfant pour les 2 premiers jetons. Si l'enfant a réussi avec beaucoup d'aide, il lui est demandé de recompter-pointer "tout seul". Les nombres à compter sont : pour les grands, 7 et, en cas d'échec 5; pour les moyens, 5 et, en cas de réussite 7, en cas d'échec 3; pour les petits, 3 et, en cas de réussite, 5.

Test cardinal : Si l'enfant ne conclut pas spontanément son comptage-pointage, par exemple en répétant le dernier nombre ou en rajoutant "il y en a x", on le soumet au test cardinal, ce test consistant à cacher les jetons aussitôt que l'enfant a terminé de les compter et à lui demander combien il y en a.

II ETUDES GENETIQUES

1) Récitation de la suite des nombres

a)Codage : On retient le nombre n maximal tel que la suite 1, 2, ..., n "récitée" par l'enfant, au cours de l'épreuve 1 ou au cours d'un comptage en cours d'entretien, est parfaite.

b)Résultats : La plupart (au moins 75%) des enfants "récitent", vers 4;3, la suite des nombres jusqu'à 4 au moins; vers 4;9, jusqu'à 5 au moins; vers 5;3, jusqu'à 7 au moins; vers 5;9, jusqu'à 11 au moins, et enfin, vers 6;3, jusqu'à 14 au moins.

2) Principes du "Comment compter"

a)Critères d'attribution : Les critères précis sont donnés dans l'étude complète (Fischer, à paraître). Précisons ici simplement que nous avons d'une part tenu compte des comptages spontanés de l'enfant, d'autre part admis un certain taux d'erreurs.

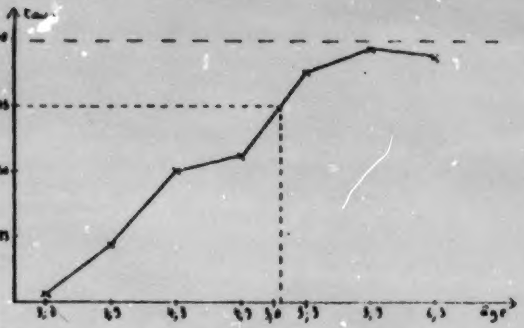
b)Tableau des attributions (en nombres) :

n :	3				5					7		
GA Principes	3,3	3,9	4,3	4,9	4,3	4,9	5,3	5,9	6,3	5,3	5,9	6,3
P_{bi} (n)	17	22	30	29	22	25	30	32	30	23	27	30
P_{ss} (n)	11	21	24	29	18	24	28	31	30	25	29	30
P_{ca} (n)	6*	13*	19	25	16*	20	29	32	30	25	31	30
Les 3 (pour n)	4	12*	19	25	16	18	28	31	30	20	25	30

* sur 31 enfants (32 pour les autres cases)

c)Représentation graphique du taux d'attribution de l'ensemble des 3 principes, pour 5, en fonction de l'âge (voir page suivante)

d)Commentaires : Les tableau et représentation graphique montrent que les progrès sont réguliers : les seules légères entorses à cette régularité concernent des âges où le taux de réussite est proche de 1. C'est vers 5;1 que le taux de 0,75, pour l'attribution de l'ensemble des 3 principes (pour 5), est atteint.



e)Remarque : Les tableaux croisés (que nous ne donnons pas) et calculs de χ^2_M ou χ^2_{Mc}(4) montrent que :

- chez les petits, et pour 3, le P_{ca} a été attribué significativement moins souvent que le P_{bi} et le P_{ss};
- chez les moyens, et pour 5, la seule différence significative qui subsiste est celle entre les P_{ca} et P_{bi};
- enfin chez les grands, et pour 7, non seulement aucune des 2 différences significatives ne subsiste, mais P_{ca} est même plus (non significativement toutefois) réussi que P_{bi} et P_{ss}.

Pour la difficulté relative entre les P_{bi} et P_{ss}, pour 3, il faut regrouper petits et moyens pour voir que le P_{bi} est significativement plus attribué que le P_{ss}, alors que chez les grands, pour 7, la tendance (non significative) est encore une fois inverse.

III HYPOTHESES SUR LE ROLE DU COMPTAGE

1) Hypothèse générale

Notre hypothèse générale est que le comptage joue un rôle important dans la dénomination des premiers nombres par l'enfant. Nous ne donnerons ci-après qu'un argument direct (observation des comptages) et un argument indirect (vérification d'une conséquence) en faveur de cette hypothèse.

a)Tableau des taux de réussite par comptage extériorisé (à l'épreuve de dénomination) : Sur le tableau de la page suivante, on voit nettement que le taux des réussites par comptage extériorisé (= comptage à haute voix, pointage du doigt ou des yeux, mouvements des lèvres,..) augmente avec la taille du nombre (une seule petite exception). Par contre la variation en fonction de l'âge est moins nette. Néanmoins en négligeant les cases à faible effectif et en tenant compte de certains facteurs expérimentaux, on peut dire qu'avec l'âge, à nombre constant, le taux des réussites par comptage extériorisé diminue.

Nombre / C.A.	2	3	4	5	6	7
3,3	.06	.50*	.30*	▨	▨	▨
3,9	.04	.70*	1.00*	▨	▨	▨
4,3	.13	.32	.67	.70	▨	▨
4,9	.17	.35	.67	.71	▨	▨
5.3	▨	.43	.48	.81	.79	.88
5,9	▨	.32	.42	.61	.79	.94
6,3	▨	.10	.20	.38	.62	.74

* moins de 10 enfants

b) Vérification de la quasi-implication[(2)] :

(savoir dénommer n) $\xRightarrow[q]{}$ (savoir compter n), pour n = 3, 5 et 7.

sait compter 3	sait dén. 3 oui	sait dén. 3 non	
oui	9	7	16
non	4	43	47
	13	50	63

Petits : i.i. = -1,83

sait compter 5	sait dén. 5 oui	sait dén. 5 non	
oui	24	10	34
non	3	27	30
	27	37	64

Moyens : i.i. = -2,71

sait compter 7	sait dén. 7 oui	sait dén. 7 non	
oui	58	17	75
non	2	19	21
	60	36	96

Grands : i.i. = -3,07

La quasi-implication est donc vérifiée dans les 3 cas.

2) Hypothèse précise

Une de nos hypothèses précises est que l'appréhension rapide des petits nombres (> 2) peut n'être que l'étape ultime d'une intériorisation du comptage. Ce sont l'analogie entre l'apprentissage du comptage et l'apprentissage suivant la théorie de Galperin (Galperin, 1979), et les résultats de ce dernier, qui nous conduisent à une telle hypothèse. En effet, dans l'apprentissage du comptage on retrouve les principales caractéristiques de l'apprentissage selon Galperin : l'orientation joue un rôle primordial, l'action accompagnée du langage (comptage-pointage à haute voix) s'intériorise peu à peu, le nombre de comptages réalisés par un enfant est souvent impressionnant (à rapprocher du nombre important d'exercices au cours de chaque étape de l'apprentissage chez Galperin). De plus, au niveau des résultats, on retrouve aussi les retours à des étapes plus primitives en cas de difficulté : nous avons par exemple recensé un nombre non négligeable d'enfants qui, lors de l'épreuve de dénomination, ont manifesté systématiquement des signes extérieurs à partir d'une certaine taille de nombre, alors que pour les nombres inférieurs ils ne l'ont pas fait. Or l'apprentissage, par exemple d'une figure géométrique, suivant Galperin, conduit finalement à une reconnaissance de cette figure par un simple coup d'oeil. Il se pourrait donc que l'intériorisation progressive du comptage conduise elle aussi à une reconnaissance (dénomination) finale du nombre en un coup d'oeil.

Conclusion : lorsque nous aurons encore précisé que les résultats ci-dessus, qui mettent à eux seuls en évidence l'importance du comptage, ont été confirmés par nos autres observations et analyses, et que, aussi bien la psychologie, très influencée par Piaget, que la didactique, avec la profonde réforme de l'enseignement des mathématiques à l'école élémentaire (1970 en France), ont eu tendance à la nier, nous pourrons conclure qu'il serait souhaitable que l'Ecole tienne davantage compte de ce phénomène important qu'est le comptage du jeune enfant.

Notes :

(1) χ^1_M désigne le test du χ^1 calculé par la méthode de Mac Nemar. Corrigé, pour effectifs théoriques insuffisants, on le note χ^1_{MC}.

Précision importante : Nous travaillons au seuil, fixé une fois pour toutes, de 0,05.

(2) On note $a \underset{q}{\Longrightarrow} b$ une quasi-implication. Si card$[E(a)] \leq$ card$[E(b)]$, où $E(a)$ [resp. $E(b)$] désigne l'ensemble des enfants ayant satisfait a (resp. b), $a \underset{q}{\Longrightarrow} b$ est admissible (à notre seuil de travail = 0,05) si l'indice d'implication, que nous notons i.i, est $< -1,64$ ou si $E(a) \subset E(b)$. [Référence : Gras, 1980].

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AN INVENTORY OF THE PERFORMANCE OF HOW KINDERGARTENERS (5 - 6 YEARS OF AGE) USE THE DIFFERENT ASPECTS OF NATURAL NUMBERS

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Dans les annés 1970, dans la discussion sur l'enseignement de la mathématique élémentaire (en Allemagne), les aspects suivants des nombres naturels ont été étudiés: aspects cardinal et ordinal, aspects de la mesure et de l'operateur, aspects de compter, du code, de la représentation et de l'échelle. Le but de cette recherche est d'obtenir une vue générale sur comment des enfants Âgés de 5 à 6 ans, avant aller à l'école, arrivent à manier ces aspects (des autres recherches ont en vue). Ainsi, 24 enfants ont été soumis à des entretiens individuels - au total six fois par enfant.

(Les résultats ne pourront être donrés qu'un juillet 1981 à Grenoble, car cette recherche n'est pas encore finie en ce moment (avril 1981).)

ASPECTS OF NATURAL NUMBERS IN ELEMENTARY MATHEMATICS EDUCATION

During the 1970's different aspects of the concept of natural numbers were elaborated within the discussion on mathematics education in primary schools (grades 1 - 4; 6 - 10 years of age) in Germany (Wittmann, 1972,1975; Steiner,1972; Freudenthal,1973; Müller-Wittmann,1977). As a result of this discussion the follwoing aspects of natural numbers have to be considered:

- ordinal aspect (example: Charles is the fifth child in his family.)
  (natural numbers as counting numbers (one, two, three, ...) or as ordinal numbers to mark a certain position in a well-ordered set of objects (the first, the second, the third, ...));
- cardinal aspect (example:Margaret has got five dolls.)
  (natural numbers as numbers for the numerosity of finite sets(numerosity numbers))
- aspect of measure (example: This tree has a length of five metres.)
  (natural numbers as measuring numbers of magnitudes relative to certain unit magnitude);
- aspect of operator (example: John has five times as much money as Richard.)
  (natural numbers as multipliers (operators) on magnitudes (once, twice, three times, ...));
- aspect of reckoning number(example: 4 + 1 = 5)
  (natural numbers as elements of an algebraic structure or as sequences of

digits which can be processed by algorithms)

- aspect of coding (example: The horse with number five won the race.) (natural numbers as signs to make differences among objects)
- aspect of representation (examples: $5_{(10)} = 101_{(2)}$. "Five" is a one digit number (in the decimal system of notation).) (natural numbers as sequences of digits relative to a certain base (positional systems of notation))
- aspect of scale (example: Today it is five degrees centigrade.) (natural numbers as marks on a scale)

In the textbooks for primary schools in Germany the cardinal aspect is located - chronologically and systematically - in the first position: The first natural numbers - 1, 2, 3, ..., 10 - are introduced as cardinal numbers (of finite sets); all the other aspects appear, too, but later on.

According to Piaget the concept of natural numbers is constituted by the coordination of the cardinal and the ordinal aspect (Piaget-Szeminska, 1941, 1975; Piaget, 1950). In particular, the conservation of number is an essential indicator that a child has reached the operational level of his cognitive development in the number concept. Beyond this, conservation is looked at by Piaget as a "necessary condition for all rational activity." (Piaget, 1975, p. 15) (For a refined criticism of the well known interpretation of Piaget's conservation tests and a new interpretation ("conflict hypothesis") look at Bryant, 1974.)

Brainerd (1973, 1979) has tried to show that the ordinal aspect is more important for the development of the number concept and more attuned to the children's cognitive development than the cardinal aspect; he does not accept the theory of Piaget either.

Mpiangu-Gentile (1975) carried out an investigation which was motivated by the question: "Does training in arithmetic concepts have a different effect on conservers and non-conservers?" (p. 184) They got this main result: Non-conservers benefited as much from arithmetic training as did conservers (between 5 and 6 years of age). For similar results for kindergarteners and first graders, respectively, look at Baroody (1979) and Pennington (1977). Mpiangu-Gentile think it more profitable "to consider arithmetic and conservation of number as conceptualizations that develop simutaneously." (p. 194)

Already in 1962 Gréco gave an interpretation of his research according to which the aspect of natural numbers as counting numbers is more than just a relative unimportant linguistic fact (as Piaget said); moreover, this aspect seems to be appropriate in also supporting the other aspects of natural numbers.

Models of number development emphasizing counting skills have been constructed by Schaeffer-Eggleston-Scott (1974) and, more recently, by Gelman-Gallistel (1978).

Whether one accepts the results of Mpiangu-Gentile (1975) or not the follwoing statement seems to be a good basis for research:"..., the effort should be directed toward a genuine evaluation of what arithmetic concepts the child knows and how well he knows them before he starts formal instruction in arithmetic. This knowledge, rather than conservation of number, should probably constitute the basis for teaching arithmetic."(p.191) The papers of Ginsburg (e.g. 1975, 1977) have shown us that pre-schoolers know a lot of informal arithmetic (see Gelman-Gallistel,1978,too). Recently Hendrickson (1979) and Comiti (1980; see 1977, 1978, too) have presented interesting papers on investigations on arithmetic abilities of incoming first graders which methodologically have much in common with the research reported here.

OUTLINE OF A RESEARCH PROJECT CONCERNING THE PERFORMANCE OF HOW KINDERGARTENERS USE THE DIFFERENT ASPECTS OF NATURAL NUMBERS

Purpose

The purpose of this study was to make a survey of the performances of how children between 5 and 6 years of age use the different aspects of natural numbers discussed above. The first reason for this purpose is in this question: Does the primary school really refer to the actual arithmetic knowledge and competencies and the individual peculiarities of the incoming children? A second but not less important motivation consists in the present research situation mentioned above: It seems to be reasonable not only to look at the number conservation but to take into account, too, that the development of the number concept may be effectively influenced by other factors (e.g. the probably mutual influences of semantic and syntactic factors). In this respect we hope to continue with further research.

Sample

12 children were chosen from a kindergarten in an urban district of Cologne and another 12 children from a kindergarten in a suburban district of Wuppertal (FRG): 15 children with at least 6 years of age on June 30th, 1981 (school beginners in autumn 1981), 6 children with 6 years of age up until to December 31st, 1981, 3 children with 6 years of age only in 1982. Each sub-sample consisted of 6 girls and 6 boys ranging on the whole scale of the cognitive development of children between 5 and 6 years of age (according to the judgements of the headmistresses of the two kindergartens).

Method

Each child was interviewed individually for six times - but only once a day. Each child was given as much time as was needed to answer; so the interviews ranged from about 15 to 35 minutes. In nearly all the interviews materials were used, the children being free to use them as they wished.

According to the purpose of this study it was the aim of these interviews

- not to compare the children under absolutely standardized conditions,
- but rather to find out how the various children would answer the questions and tasks concerning the different aspects of natural numbers.

Above all we were interested in making a survey - first - on the range of the answers children of this age may give and - secondly - on the peculiarities which may appear among such children.

The interviews were conducted in a room of the familiar kindergarten (March 1981 at Cologne, May 1981 at Wuppertal). They were recorded by a video tape recorder; this did not seem to influence the children's performance at all.

Procedure

Each interview situation was introduced by presenting some materials (e.g. chips, Cuisenaire rods, coins etc.) and by a question or an instruction. We give an example [addition, nn; look at next page]:

You have got a box [open!] with black chips.

I have got a box [open, too!] with white chips. [Both boxes then were put side by side.]

Put into the empty [third open] box as many red chips as we both have got together!
The answer of the child was followed by one of the following questions if the behaviour of the child did not clearly show his tactic: How do you know that? - How did you work it out?

Owing to lack of space we can only sketch the content areas of the interviews in a table:

	cardinal aspect, nn (not using numerals)	cardinal aspect, n (using numerals)	ordinal aspect	aspect of magnitude	aspect of operator	aspect of scale
essential conceptual components	ess. conc. comp., nn	ess.conc. comp., n	ess.conc. comp.	-lengths -weights -money -spans of time	-lengths -weights -money -spans of time -cardinal numbers	ess. conc. comp. (points of time)
>-relation (more, greater,...)	>-rel.,nn	>-rel.,n	>-rel.	>-rel. (as above)	>-rel. (money, spans of time)	>-rel.
addition	addition, nn	addition, n	addition	aaddition (as above)	--	--

Counting and knowledge of the use of numbers:

- Please,count out loud for me as far as you can!
- writing and reading numerals (written with digits in the decimal system)
- Where do we use numbers?(incl. a series of key-words)

Results and Discussion

Results and the discussion will be given at the conference at Grenoble in July 1981, since at the time of writing this paper (April 1981) only the Cologne sub-sample has just been interviewed.

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CONSTRUCTIVISM, THE TEACHING EXPERIMENT, AND MODELING

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Le but principal de cette communication est de décrire la méthodologie utilisée dans le projet Interdisciplinary Research on Number (IRON) pour l'élaboration d'un modèle de l'apprentissage mathématique des enfants. En accord avec Piaget, l'apprentissage est interprétée comme reconstruction active de la connanissance face à l'expérience. L'importance didactique du projet se situe dans la tentative d'expliquer comment les enfants construisent le nombre et les influences que des interventions directes peuvent avoir sur ce processus de construction. La formulation d'explications appropriées a exigé une observation longue et intensive de la résolution de problèmes par des enfants dans une expérimentation didactique. Cette méthdologie diffère de l'expérimentation didactique soviétique puisqu'elle est employée dans le but de construire des modèles d'apprentissage plutôt que dans l'élaboration de programmes. Par "modèle" nous entendons l'ensemble de constructions théoriques qui expliquent l'interprétation de l'enfant dans sa résolution de problèmes. Puisque le modèle essaie d'expliquer la tâche de l'enfant, il y a aussi une tentative d'inférer la compréhension de celui-ci. Cela, en combinaison avec l'accent sur l'étude de la reconstruction cognitive, aboutit à un modèle qui a des implications importantes en ce qui concerne les premieres années de curricula sur le nombre. Ce modèle n'est, pourtant, intéressant aux éducateurs de mathématiques que si les programmes qui vont incluve ces implications fonctionnent--s'ils représentent une amélioration dans la qualité de la pédagogie mathématique.

Steffe, von Glasersfeld, Richards, Cobb, and Thompson[1] (1981) have recently proposed a model which, it is claimed, constitutes an explanation of how a child might construct number. The primary purpose of this paper is to describe the methodology used in formulating this model. The underpinning epistemology and purpose of IRON will be considered to provide a context

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[1] The abbreviation IRON (Interdisciplinary Research on Number) is used throughout this paper to stress that neither the model nor the methodology can be attributed to a single individual or discipline.

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for discussion of the methodology. In the course of this analysis, the appropriateness of the constructivist teaching experiment for formulating explanations of children's mathematical learning will be indicated. Limitations of space preclude a discussion of the model itself. However, the paper by Steffe (1981) exemplifies the type of explanation the constructivist teaching experiment yields.

CONSTRUCTIVISM

Modeling. The members of IRON believed that one could go beyond observable behavior and attempt to explain children's thinking. Obviously, a child's thinking is not accessible to inspection or description. The explanation will therefore have to be in the form of models. A model, as the term is used in this paper, is a constellation of theoretical constructs which explains the child's observed behavior. As a model can never be compared with the phenomen it purports to explain, it can never be claimed that it is a replica of or is isomorphic to the child's cognitive mechanisms.

Implicit in any transition from observable behavior to a psychological model is the structuralist assumption that behavioral regularities are an observable manifestation of underlying psychological regularities. The process of constructing a theory of the child's acquisition of a concept involves identifying regularities in behavior and building a model which accounts for these regularities.

Epistemological assumptions. Following Piaget (1968, 1976), experience was considered to be essential for the construction of any concept. Piaget (1968, p. 147-150) transcended behaviorism with its emphasis on genesis (change) without structure and preformism with its emphasis on structure without genesis. He reasoned that, on the one hand, genesis emanates from a structure and culminates in another structure. On the other hand, there is no absolute beginning to the sequence of structures, although the psychologist normally stops at birth. In short, structure is not subordinated to genesis or vice versa. From this perspective, learning is characterized as the restructuring of existing knowledge in the face of experience.

PURPOSE

Members of IRON decided to work together for a variety of reasons which reflected their widely differing disciplinary backgrounds. The mathematics educators involved in the project had been influenced by Dewey and Brownell as well as by Piaget when formulating their position concerning the psychology of number. For example, Dewey after conducting conceptual analyses of the concepts "whiteness" and "number," declared that "number is a rational process, not a sense fact" (McLellan and Dewey, 1895/1908, p. 23). Brownell came to a similar conclusion.

> Neither does nature provide the child with tangible evidence of number which he can apprehend immediately and thus come easily to know through sense perception. There is no concrete quantity "five-ness" in five dogs ... Neither is there any "five-ness" in ":.:", or in "five" or in "5". In each case the "five-ness" is the creation of the observer; it is a concept or an idea which the observer imposes upon the objective data (Brownell, 1935, p. 20).

Piaget's conceptual analysis of the concept of unit led him to note that "Elements are stripped of their qualities and become arithmetic unities" (1952, p. 37). However, neither Dewey, Brownell, nor Piaget explained how children construct number. Dewey's analysis was primarily biogenetic, as one would expect of a pragmatist. Brownell produced a hierarchy of behaviors but did not attempt to construct a model while Piaget did not explain how the synthesis of classification and ordering structures results in elements being stripped of their qualities. The members of IRON wanted to transcend Dewey's, Brownell's, and Piaget's work and formulate an explanation of both how children might construct number and how this process may be influenced by direct intervention.

METHODOLOGY

Identification of behavioral regularities. A methodology which was compatible with the purpose mentioned above would involve observation of and interaction with children over an extended period of time. A series of Piagetian clinical interviews would not constitute a suitable methodology because the researcher would not intentionally intervene and attempt to precipitate cognitive change in the child. The teaching experiment methodology, on the other hand, did allow direct intervention as well as longitudinal observation. The constructivist teaching experiment should, however, be distinguished from that used in the U.S.S.R. The latter involved the construction of a curriculum before the experiment was conducted. The experiment was then performed to see whether or not a particular curriculum "worked" (Menchenskaya, 1969; Kantowski, 1978). In short, the Soviet's objective is to build a curriculum per se. The reason for conducting a constructivist teaching experiment, on the other hand, is in harmony with Vygotsky's original assessment of the methdology--its essential function should be the production of models.

The constructivist teaching experiment involves a series of teaching episodes interspersed with occasional clinical interviews. The clinical interviews give the researcher the opportunity to build detailed models of children's thinking at particular moments in their development. On the other hand, "it is the teaching episode which allows us to study the constructive process--those critical moments when restructuring takes place as evidenced by alterations in the child's behavior" (Steffe and Richards, 1981). While conducting a teaching episode, the teacher-researcher "intends (1) to test the limits of a model he has of the child's knowledge with regard to particular content structures and (2) to investigate how various components of that model may change under the pressure of direct interference (Steffe and Richards, 1981)". This involves "(1) interpreting what he or she (the researcher) sees the child doing in terms of a model and (2) attempting to perform the ultimate act of decentering by conceiving of his or her own actions from the child's own perspective (Steffe and Richards, 1981)."

Procedure for constructing models. It was assumed that any child's observed behavior was not produced randomly; the child's behavior was the observable manifestation of an intentional act. This assumption is compatible with Smedslund's statement that

> I gradually came to realize that the only defensible position is always to presuppose logicality in the other person and always to treat his understanding of given situations as a matter of empirical study. From this point of view, people are always seen as logical (rational) given their own premises, and hence behavior can, in principle, always be understood. This also applies to small children. (Smedslund, 1977, p. 3)

The child's intentions or premises were to be captured by inferring his or her conceptualization of the task. That is, the researcher poses the question, "Given that the child intended to produce the behavior I have observed, what was the task the child was attempting to solve?" In essence, the researcher attempts to work backwards from the observed behavior to the task the child established. This "working backwards" approach is viable because, following Piaget (1968, 1976), it was assumed that the "building blocks" out of which the child constructs his or her task are either re-presentations of sensory motor activity or conceptual operations. The child whose conception of the task takes the form of re-presentations of sensory-motor activity attempts to materialize this activity when solving his or her task. Conceptual operations, on the other hand, are thought of as the results of reflective abstractions from sensory-motor activity. The sensory-motor activity is said to be implicit for the child. In solving a task, the child may have to attempt to materialize the activity implicit in some of the operations. For example, a child who has constructed the abstract concept of number may conceptualize the task $12 + 7 = \_\_\_$ as requiring the specification of the numerosity which is itself composed of numerosities denoted by "12" and "7", but yet count in solution. In effect, the child may have to drop down a level of abstraction when solving the task he or she has constructed. The researcher attempts to infer the re-presented sensory motor activity or conceptual operations which constituted the child's task from the observed sensory-motor activity.

THE CONSTRUCTIVIST TEACHING EXPERIMENT AND MATHEMATICS EDUCATION

A fundamental question for mathematics educators is, "What is the child's conception of number and what can be done to help the child make the constructive journey to higher levels of understanding?" It is claimed that the constructivist teaching experiment allows the researcher to address this question. On the one hand, the objective is to study cognitive restructuring. On the other hand, by modeling the task the child constructs, an attempt is made to infer the child's understanding. Therefore, such models have substantial implications for early number curricula which attempt to be consistent with the ways children learn. However, "The value of a knowledge of psychology in general, or of the psychology of a particular subject, will be best made known by its fruits" (McLellan and Dewey, 1896/1908, p. 1). In short, IRON's theory will only be of value to mathematics education if it works--if employment of instructional techniques derived from the theory represent an improvement in the quality of mathematics education.

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QUERIES AROUND THE NUMBER CONCEPT

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There are a lot of unexplained phenomena around number such as conceived by young children (kindergarten and lower grades). Their ideas on numbers strongly diverge from ours. How can we get a hold and understand the children's ideas and how apply this knowledge in instruction?

How can we get a hold on this ideas?
I will first tell you something about our method of investigation, which we called that of 'intensive *mutual* observation'.
Than I will give a survey on the phenomena that struck us and which are the subject of our research.
Finally I will enter into details on three of these subjects:

a) *acoustic counting;*
b) counting *movements* rather than objects;
c) the number concept described in *psychological* rather than mathematical terms.

1 *Mutual observation*

In talks with children (K-2) on certain themes we try to get a hold on their ideas. If, for instance, second graders have figured out ☐ - 9 = 24, we like to know how each of them found the solution. Many children, however, give a cliché answer:

'Cause',
'I just thought it',
'I did it in my head',

while cheerfully laughing. But one cannot get a hold on what they actually did. C. do the children not understand what one wants to know? Or isn't this one way observation of children basically wrong? Let me illustrate this. During my talks with the children I note down as much as possible. What does the pupil tell me? What does it do? But noting down creates long waiting periods for the pupil. I tried to break this silence by *reading aloud* what I wrote. It created a relaxed sphere, I thought so.

In fact more was happening:

I the children notice that what one writes down, regards them. Most of them had never hit on this ideas before. They are proud that all they say is noted down;

2 they correct if something is noted down erroneously. It is a nice thing to provoke this once intentionally;

3 while trying to help you recording, they better realise what they did and what they did not;

4 they reflect on their own thoughts so they take their place as it were beside us as co-observers;

5 in this way they grasp what we want to know about them.

This, indeed, is the big problem: we ask them 'why' and 'how' but this does not at all mean that they know what we want to know. But by *reading aloud* what one has recorded during the observation, the observed one gets a good chance to recognise himself in the record and to possibly make emendations on it. This seems to me the fundamental distinction between the one-sided observation of machines, animals, systems and the mutual observation of persons.

2 *Quries around the number concept*

Though there is more to the philosophy of this method of investigation, I leave this subject in order to deal with subjects of content.
The title of this talk was 'Queries around the number concept'. Here they are:

1 Acoustic counting.
Is counting always tied to counting sets, or doesn't it play an independent acoustic part in the children's life?

2 Counting movements.
Do children count objects or rather the movements of showing to the objects?

3 Psychology and number concept.
Why is the number concept of children mostly described in terms borrowed from mathematics? Is this right? Isn't there a task for psychology?

This three subjects will be tackled afterward. Meanwhile we continue our list:

4 Dynamic notations.
As soon as situations and ideas are noted down by means of symbols, they become less manageable for the children, so it seems. There are notations that avoid this curtailing influence on the children's thought. We call them dynamic notations.(Nesher, Katriel, 1979).

5 Completing, substituting and conserving are typical activities of children in mathematics. Completing, for instance, is a basis for adding and subtracting; substituting numbers by numbers, situations by situations is an important mathematical activity.

6 Adding and subtracting need not be converses of each other in many situation.

7 Finally themes of investigation can be found in the use of hand-hold computers.

Three of these subjects will be considered more closely:

1 Acoustic counting:
Many kindergarten teachers think that counting is counting sets of objects such as it is taught in the first grade. However a young child needs no objects to show it can count. It can cpunt merely acoustically, indeed.

> *Marlieke (three years old) walks on the trace made by a tractor and spontaneously starts counting '4, 8, 2, ...'*

Another example of acoustic counting (Van den Brink, 1980):

> *Paul (three years) is in the wood with a group of older children. They are jumping over a ditch. '1, 2, hop.' Paul too. Ready to jump he counts: '7, 8, 9, 10, 11, 12', runs away, gets the other side, and smiles happily.*
>
> *Bernt (five years) shows another characteristic of acoustic counting. He counts from 1 to 29 and than continues with: 'twenty-ten, twenty-eleven...'*

These examples show that counting is not only counting sets. Children can make you aware of other functions of counting. Acoustic counting is on the one hand closely tied to rhythm and movement. Numerals support or accompany movements (as in the case of Marlieke) or instigate movements (as in the case of Paul). On the other hand while counting children are directed on sound systematics (Bernts inventing new numerals).
For instruction in kindergarten this means that the teacher can stimulate counting by all kind of counting songs, movement games, counting out and that she should certainly not use counting only to count sets.

2 This brings me to the second domain of research: counting movements.
It is an activity where children make a lot of 'mistakes' - in quotation marks: skipping, repeatedly counting, not exhausting and suchlike (Gelman, Gallistel 1978; Mierkiewicz, Siegler 1980).

> *Cindy (five years) counts 7 blocks arranged in a row in front of her. She does it in this way:*
>
> *1 2 3 4 5 6 7 8 9 10*
> □□□ □ □ □□
>
> *'Once more', I ask her. Then she counts*
>
> *1 2 3 4 5 6 7 8*
> □□□ □□□□
>
> *'How is it possible? First there were ten and now eight?'*
>
> *'Yes, one less', she says, starts anew and eventually indicates all blocks correctly while counting:*
>
> *1 2 3 4 5 6 7*
> □□□□□□□
>
> *'Now she has learned it', I think, 'she is ready.' But she continues and again skips one:*
>
> *1 2 3 4 5 6*
> □□□□□□□
>
> *And then:*
>
> *1 2 3 4 5*
> □□□□□□□
>
> *'What do you do now?', I asked her surprised. She answers: 'Just skipping one.' She continues counting. Not the blocks but her movements.*

Cindy does not mind skipping, and this is what happens more often: at hop-scotch, skipping with the rope.

If one gives kindergarten-children tasks of counting back they manage by a kind of skipping. They again and again use the same counting sequence 1, 2, 3, 4, 5, 6, 7 while stopping one step earlier: 1, 2, 3, 4, 5, 6 ; 1, 2, 3, 4, 5, ... Children's counting connects numerals to movements before it does so to objects.

One more example:

> *Marije (first grade, seven years) must count a large number of pencils. She arrives at 20 while showing to a pencil. Yet while pronouncing 21 (Dutch: één-en-twintig) she shows to three pencils corresponding to the sound pattern: één-en-twintig.*

There are strong arguments for the case that children do not count objects but numbers of times they *indicate* objects:

1 firstly the numeral is always pronounced immediately *after* showing to some object;

2 the movement of indicating is created at a certain moment whereas the object is constantly present. So there is a stronger bond between the numerals with the counting act than there is with the object

If we admit that children count movements rather than objects, one can understand why they:

3 skip (as Cindy did);

4 count repeatelly;

5 do not exhaust.

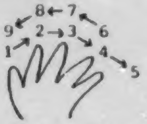
Indeed what matters to them are movements rather than objects. Acoustic counting aims exclusively at movement and sound systematics, with no objects concerned. So it seems natural that counting objects is performed as sort of application of counting acoustically. This means as it were children are *balancing* between different kinds of countinh: acoustic counting that accompanies movements and counting influenced by the presence of objects. Mistakes made by children (skipping, repeatedly counting, not exhausting) can be understood in this way.

For instruction this means that children should be made aware of these errors by means of conflict or surprise.

Two examples:

a) while the teacher counts, the child shows to beads at a string After one turn around the string she simply goes on, while the children continue showing...;

b) each pupil knows that if one count twice the fingers of a hand, it is 10. Now I count:



and it is 9.

'You skipped one.'

'You did not count well.'

'One more.'

By means of such conflict situations we draw their attention to such mistakes as skipping and repeatedly counting.

3 Finally we will discuss the problem whether the number concept should be investigated and described only in terms borrowed from mathematics:

cardinal number, ordinal number, registration number, calculating number, measuring number. An example:

> *Bob (seven years, 2nd grade) is given the problem 34 + 4 = I am surprised that first he counts from 1 to 34 and then continues four steps. Other children start as late as 30.*

Each pupil in his own way operationalises the number 34. Essentially the symbol 34 is the extreme abridgment of *individually determined procedures* which children apply in certain situations. The number is not associated with sets but with situations and procedures. Therefore it could be important in psychology to speak about *situation bound numbers*. So we could for instance distinguish and investigate: the bus number, the number in juggling-tricks (Van den Brink, 1980) and the age number.

The age number plays a particularly important part in age mixed kindergarten classes. Little children sometimes as a joke use age numbers as though it were cardinals:

> *Kikkie-Nanje (5 years) gets one shoe only, the other is still lacking. Se says: 'But I am two years.'*

While showing them a box of paperclips I asked a group of firstgraders: 'I take a handful and you take a handful from the box. We are going to count them. And then we look who is older, you or I.' None of them found it a strange proposal. They most often took a full hand of paperclips, whereas I always made efforts to take less. After counting each of them had a feeling of victory (they were older than I am) but at the same time they understood the joke. Clearly young children can easily work with their ages as though they were simple cardinals. Colloquial language may favour this. In 'I am six' the number six means 'six years'. Moreover if numbers are interpreted as ages children can do quite a lot with them.

From acoustic counting we know that counting back is an impracticable task for young children. But Kikkie-Nanje (5 years) says:

> *'I am 5 and next 6.' I ask: 'What was you first?' 'Four', she says and continues: 'And when I was 4, I was 3. And when I was 3, I was 2. And then one.'*

Counting back succeeds because the real situation of ages is in view. For that matter teachers reinforce age numbers functioning as simple cardinals. Connections are being made between age 5 the number of candles on the cake, the number of birthday songs, the numbers of knots on the birthday cap.

There are a great many of such one-to-one connections, though some of them must be bizarre in the ages of the children. Indeed, what have 5 candles to do with the fact that somebody becames 5 years old. For the children these must be magic connections. Freudenthal calls this the magical context (Freudenthal, 1980). The age number is as it were explicitly lifted out of its situation; the 'abstract' years are symbolised by concrete objects such as candles, songs and knots.
The case with children use their abstract age number in a joke is destroyed by the blunt seriousness of stressing the cardinality in instruction. Is this not a task for psychological research, rather than by mathematical terms such as cardinal, to investigate number in contexts, situations and procedures?

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SEMANTIC CATEGORIES OF WORD-PROBLEMS, RECONSIDERED

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Cet article propose quatre stades de développement chez l'enfant dans la résolution des problèmes de calcul simple posés en langage courant. Les quatre stades sont décrits en terme de deux sous-systèmes: (1) La croissance de la complexité sémantique du langage courant de l'enfant, et (2) la disponibilité de structures logico-mathématiques supplémentaires. Dans chacun des sous-systèmes, le développement consiste en deux aspects, comme cela a déjà été observé auparavant: relations de classe et d'ordre. Nous allons décrire le développement de la sémantique du langage courant e commensant par des schémas pour former des ensembles, leur ajouter ou leur enlever des éléments, et en arrivant au stade auquel l'enfant a la capacité de manipuler des relations entre deux ensembles dans le cadre du schéma partie/partie/tout. Parallèlement nous décrivons un développement dans les structures logiques et arithmétiques de l'enfant en commensant par compter les éléments d'un ensemble pour trouver sa cardinalité et en terminant par la coordination d'inégalités et d'égalités ainsi que par las reversibilité de l'addition et de la soustraction. Nous allons interpréter en terme des quatre stades de la connaissance les découverts empiriques sur la capacité des enfants à résoudre les problèmes: "combiner", "changer", et "comparer" dans lesquels on prend en considération la position de l'inconnue.

Since the beginning of the century (Frege, 1884, Russell, 1919), it has been accepted that the concept of number has two fundamental components: class and ordinal relation. Piaget (1952) analyzed the development of these concepts in children's performance in a variety of tasks requiring analysis of set relationships and ordered sequences. In this paper we analyze arithmetic word problems that are solved by single addition and subtraction operations. We propose a semantic analysis in which meanings of problems are structures that include class and order relations. The different kinds of problems differ in the complexity of semantic structures and the operations required to derive the meaning structures from the problem texts. We postulate representational processes in children's understanding of problems corresponding to the derivations in our semantic analysis, and thereby explain the relative difficulty of different kinds of word problems. The meaning structures also can be viewed as semantic interpretation of formal arithmetic sentences. This provides an analysis of children's achievement of more sophisticated understanding of arithmetic concepts and relationships.

Previous analyses of word problems have identified four semantic categories or problems: Combine, Change, Compare, and Equalize. (These categories have been agreed by Carpenter, et al (1981a, 1981b), Heller et al, 1978, Nesher et al (1978, 1981), Riley, et al (1981), Vergnaud (1976, 1981), with minor variations. When empirical data were collected according to the above semantic categories it was agreed that Change problems are the easiest, Combine problems are next in difficulty and the Compare problems are most difficult (Nesher, 1978, Riley, 1981). It was noticed, however, (Carpenter et al, 1981, Nesher, 1978, Riley et al, 1981, Vergnaud, 1981) that a further differentiation within the semantic categories according to the position of the unknown results in a better prediction concerning the children's performance. Table 1 presents the differentiated categories.

Table 1

Title	General Description	
Combine 1	question about the union set (whole).	
Combine 2	question about one subset (part).	
Change 1	increasing,	question about the final state.
Change 2	decreasing,	question about the final state.
Change 3	increasing,	question about the change.
Change 4	decreasing,	question about the change.
Change 5	increasing,	question about the initial state.
Change 6	decreasing,	question about the initial state.
Equalize 1	question about change from smaller to larger set.	
Equalize 2	question about change from larger to smaller set.	
Compare 1	mentioning 'more'	question about the difference set
Compare 2	mentioning 'less'	question about the difference set
Compare 3	mentioning 'more'	question about the 'compared'
Compare 4	mentioning 'less'	question about the 'compared'
Compare 5	mentioning 'more'	question about the referent
Compare 6	mentioning 'less'	question about the referent

Our explanation for young children's performance on the above 16 categories is based on hypotheses about semantic schemes that are available at different developmental levels. We assume that there are two structures of knowledge that are involved in solving simple arithmetic word problems: (a) the child's knowledge of the world, and (b) the child's knowledge of logical-mathematical structures. As Piaget (1952, 1967, 1970) noted, the sources of these two knowledge structures are not the same, even though they do not develop in isolation from each other. We therefore will describe levels of development in children's semantic schemes regarding two distinct knowledge structures: Ordinary world knowledge (empirical knowledge in Piaget's terms) (Lo) and Arithmetic knowledge (logico-mathematical knowledge) (La).

Developmental Levels The description in what follows will be in terms of growth of specific semantic schemes in empirical and arithmetic knowledge, which will be sufficient to explain the children's performance in solving word problems in arithmetic. Our analysis is consistent with Piaget's theory, but articulates hypotheses about the schemata more specifically. Each developmental level will be described along the following dimensions:

Lo-The development of empirical knowledge and ordinary language.
La-The development of arithmetic knowledge and formal language.
G-The logical representation of Lo and La knowledge.
S-The underlying semantic schemes for Lo and La available to the child as schemes of action.

(Due to scope limitation 'g' and 'S' will be presented at the conference.) Each level assumes the presence of the knowledge described in the previous level. The development will be described here informally, but see Nesher, Greeno and Riley, 1981 for a more formal presentation.

Level 1 Semantic structures include reference to sets and simple operations such as adding members to a given set, removing members, and forming a new set. Sets can be identified by a variety of verbal descriptions (generic names, locations, points in time, possessors) and verbs denoting change in location or possession, such as 'put,' 'give,' and 'take' are understood. The arithmetic level consists of the ability to count and find the cardinal number of a given set.

Level 2 Semantic structures of Lo include reference to the amount of change needed to transform a set into a larger or smaller set. This is related to ability to link events by cause and effect, to anticipate results of actions described in ordinary language, and to understand sequences of events ordered in time in a unidirectional and nonreversible manner. In arithmetic at this level there is understanding of addition and subtraction operations as procedures to follow. '+' and '-' are distinct and not related.

Level 3 A scheme of part-part-whole relations is available and can be used to represent partial information with a slot for the unknown quantity. The scheme enables reversible inferences about set relationships, including the amount of difference between two specified sets. It is related to understanding of class inclusion and ability to quantify the same extension of objects according to a shift in the predicates that describe them intentionally. In arithmetic, at this level, the additive structure is reversible and includes the equality relation including the necessary inference that if a + b = c, then c - b = a or c - a = b.

Level 4 The scheme for non-symmetrical relations (which started at Level 2, in the description of a change, or comparison) is now available in a reversible manner. Directional ordered descriptions (i.e. 'more'/'less') can be handled in a flexible manner. A set can be induced by means of relative comparison. In arithmetic this level will include the ability to handle inequality, and its relationship to equality (equalizing it by addition or subtraction operation): If A > B, then A - C = B or B + C = A.

We will show that the above developmental levels can account for the levels of performance in arithmetic word problems as was found in various empirical studies: Carpenter et al, 1981; Nesher, 1978; Riley, 1981; Vergnaud, 1981. (The data will be presented at the conference.) We suggest that the four levels described above can predict the various levels of success in solving addition and subtraction word problems as detailed in Table 2. Moreover, we suggest that Table 2 is also an analysis of the 16 categories of addition and subtraction word-problem in terms of the semantic and logical structures that underlie successful performance in each of them.

Table 2

Type of Problem	Level 1	Level 2	Level 3	Level 4
Combine 1	X			
Combine 2			X	
Change 1	X			
Change 2	X			
Change 3		X*		
Change 4		X		
Change 5			X	
Change 6			X	
Equalizing 1		X		
Equalizing 2		X		
Compare 1			X	
Compare 2			X*	
Compare 3			X	
Compare 4			X	
Compare 5				X*
Compare 6				X

*In some empirical samples (Nesher, 1978) these problems fall in an earlier level, respectively.

Our semantic analysis shows how simple cognitive structures may be sufficient for some problems in a category but not for others; for example, that Change 1 and Change 2 problems can be solved using only simple operations on sets, but that Change 3 and Change 4 problems require understanding of quantitative change. According to our hypothesis, Change 5 and Change 6 require restructuring; the components are given along a time-axis, and must be represented in a non-temporal scheme. The part-part-whole scheme is sufficient for some Compare problems where one of the sets is divided into two parts, one of which matches the other specified set. A further cause of difficulty for Change 5, Change 6, Compare 5, and Compare 6, is a contradiction between the direct semantic interpretations of expressions in the problem texts ('getting more,' 'having more,' and their opposites) and the arithmetic operations (subtracting or adding) needed to solve the problems. To cope with these contradictions, children require cognitive structures that enable reversible reasoning based on underlying semantic relations.

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THE DEVELOPMENT OF NUMBER REPRESENTATION IN CHILDREN

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Nous proposons une théorie du développement de la représentation du nombre chez l'enfant au cours des années de l'école maternelle et les premières années de scolarité. La théorie se base sur des programmes d'informatique qui simulent les comportements décrits dans de multiples expériences sur la compréhension des mathématiques chez l'enfant. Avant l'entrée a l'école, la représentation du nombre se développe à force de compter. Le résultat est que la liste des nombres se transforme en une représentation dans laquelle chaque position dans la liste a le caractere d'une quantité. Au cours des premières années a l'école, un schéma Partie/Tout s'applique aux nombres; c'est à dire que les nombres sont représentés comme composés d'autres nombres. Le schéma Partie/Tout reste à la base de la capacité de résoudre des problèmes posés en langage courant et permet à l'enfant de découvrir des processus de calcul mental très efficaces. Au cours des années suivantes, lorsque la notation décimale est introduite à l'école, les nombres sont représentés, toujours suivant le schéma Partie/Tout, comme composés des dizaines et des unités. Ceci permet alors à l'enfant d'inventer des processus de calcul assez complexes. Cela lui donne aussi une base pour comprendre l'addition et la soustraction avec retenue.

Work on the psychological processes involved in early school arithmetic has now cumulated sufficiently that it is possible to construct a coherent account of the changing nature of the representation of number over the preschool and early school years. This paper outlines a theory of number representation for three broad periods of development: (1) The preschool period, during which counting and quantity comparison competencies provide the primary basis for inferring number representation; (2) the early primary period, during which the invention by children of sophisticated mental computational procedures and the mastery of certain forms of story problems points to two important expansions of number representation; and (3) the later primary period, during which the representation of number is modified to reflect knowledge of the decimal structure of the counting and notional system.

NUMBER REPRESENTATION AND ARITHMETIC PERFORMANCE PRIOR TO SCHOOL

Counting. At least two extensive studies of counting in preschool children (Gelman and Gallistel, 1978; Fuson, in press) now make it clear that preschool children possess extensive, if still incomplete (Comiti, Note 1) knowledge of the principles of counting as a means of establishing the quantity of a set. Greeno, Gelman and Riley (in press) have developed a computer model that simulates the observed counting performances. At the core of the model is an ordered list of numerons linked by a successor (next) relationship. The program establishes quantity by uniquely linking each object in the set with one of the numerons and then designating the last numeron named as the number in the set. It seems likely that through extensive practice with counting as a method of establishing quantity, the numeron list is gradually transformed into a representation in which each position in the list comes to stand for a quantity. Figure 1 displays this early representation of number. Children's capacity to "subitize"(Klahr and Wallace, 1976) small quantities is represented by the links between the smallest numerons and the set displays.

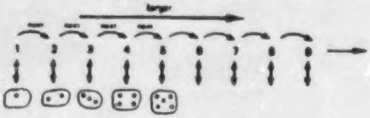


Figure 1

Quantity comparison. Experiments on quantity comparison permit further inferences about preschoolers' number representation. In these experiments two "target" numbers are named, and the subject is asked to decide which is larger (Sekuler and Mierkewicz, 1977; Robinson, 1981). Five-year-old children, like adults, take longer to make the comparison the closer together the numbers are. This "split effect" suggests that the number representation has analog features that allow direct--essentially "perceptual"--comparison of positions on a "mental measuring stick" (cf. Potts et al., 1979). This implies two further features of the child's number representation: (1) a marking of numbers later in the string as "larger": and(2) an ability to directly enter the positional representation for a number upon hearing its name (i.e., without counting up to it).

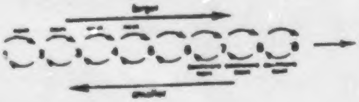
Informal arithmetic performances. With the number representation sketched it is possible to account for most of the informal arithmetic performances of preschoolers. Inspection of the various informal arithmetic performances reported by Ginsburg (1977), for example, reveals that the problems tend to be solved by counting, using tally marks, fingers, or other counting objects when necessary. This limitation to forward counting procedures is why certain classes of tasks (e.g., problems in which the unknown is in one of the subsets), are so difficult for preschoolers.

MORE SOPHISTICATED ARITHMETIC PROCEDURES IN THE EARLY SCHOOL YEARS

The methods that children use for simple mental arithmetic change during the first years of school. The new methods, together with improved ability to perform certain classes of story problems, signal an expanded representation of number and number relationships.

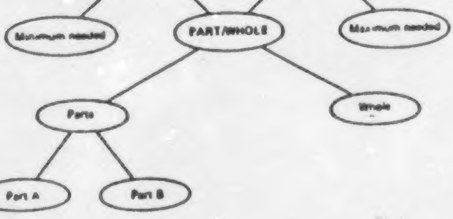
From the age of 7 or 8, children perform single-digit mental subtraction by either counting down from the larger number or counting up from the smaller number, whichever will require fewer counts (Woods, Resnick and Groen, 1975; Svenson and Hedenborg, 1979). Counting down requires a backward next relationship, and perhaps a "less" directional marker on the string of numerons (Figure 2).

Figure 2



Further, children's willingness to count in either direction must reflect some knowledge--however informal--of the complementarity of addition and subtraction. It seems likely that the discovery of complementarity is mediated by a Part/Whole schema (Figure 3), which specifies that a quantity can be partitioned, as long as the original amount is preserved. In the triple 2, 7, 9, for example, 9 is always

Figure 3

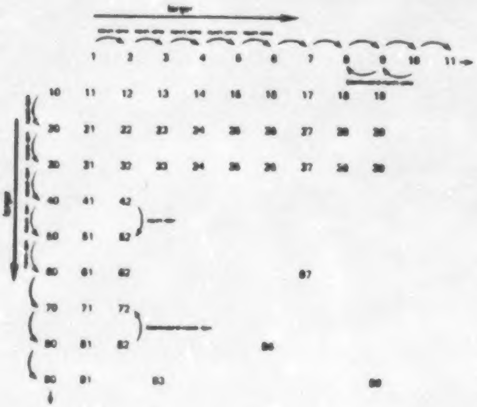


the whole, 7 and 2 always the parts. This is true whether the problem given is 7 + 2 = ?, 9 - 7 = ?, 2 + ? = 9, ? + 7 = 9, etc. The schema, since it specifies relationships between quantities rather than a procedure for operating on quantities, permits either the counting down or the counting up method of solution.

The Part/Whole schema also plays a role in successful performance of story problems. Recent work by Riley, Greeno, and Heller (in press) provides a family of formal computational models that account for the pattern of development of story problem competence found by a number of investigators (e.g., Vergnaud, in press; Carpenter and Moser, in press; Nesher, in press). The models make it clear that it is application of Part/Whole that makes it possible to solve difficult classes of story problems (for example, problems with the unknown in the starting set) that children usually cannot solve until the second or third year of school.

DEVELOPMENT OF DECIMAL NUMBER KNOWLEDGE

The next important change in number representation occurs when the decimal structure is learned and numbers are reinterpreted in decimal terms. Figure 4 sketches the kind of number representation that appears to be characteristic of children who would be judged to "understand" the decimal system. Along the rows a "next-one" relationship links the numbers. This can be extended indefinitely, as shown in the top row, indicating that a units representation of number co-exists with a decimal representation. Along the columns a "next-ten" relationship links the numbers. In a fully-developed number representation this "next-ten" link would hold for the numbers inside the matrix as well as those along the edges, permitting more efficient addition or subtraction of the quantity 10 than of other quantities. The most important feature of this new stage of understanding is that each of the numbers is represented as a composition of a tens value and a units value. This means, in effect, that two-digit numbers are interpreted in terms of the Part/Whole schema, with the special restriction that one of the parts be a multiple of 10.



The acquisition of place-value concepts has been much less studied than has the acquisition of simple number concepts. Recent work in our laboratory, however, provides initial evidence for this kind of fundamental restructuring of number representation as place value knowledge is learned. Evidence from three classes of tasks is presented here.

Quantity comparison. A relatively early indication of a developing decimal number representation is children's ability to compare two quantities on the basis of the tens value only, without reference to the units value. We have observed this performance in tasks in which written digits are compared, Dienes block displays are compared, and written digits are compared with block displays. Virtually all of the second and third graders we have studied showed this ability.

Mental Arithmetic. In another study we have asked seven-year-olds to mentally add a single-digit and a double-digit number. Both reaction times for solutions and children's reports of their solution methods were analyzed. About a quarter of the children studied used a strategy in which they broke up the two-digit number into a tens component and a ones component, then recombined the tens component with whichever of the two units quantities was larger. This produced an optimally efficient "counting on" solution. For example, for 23 + 9, the mental "counter" was set at 29 and then incremented 3 times to a sum of 32. This procedure depends upon a recognition that the whole 23 is made up of two parts, 20 and 3. The total sum, then, has three parts: 20, 3, and 9. The Part/Whole schema allows the calculation to proceed in a manner consonant with the laws of both commutativity and associativity $(23 + 9 = (20 + 3) + 9 = (20 + 9) + 3)$, although there is no hint that the children have any formal awareness of these laws.

Regrouping in subtraction. In a series of studies on the acquisition of subtraction, we have been able to observe, and build computer programs that model in detail, several stages in children's understanding of written borrowing and (its analog) trading of Dienes blocks (Resnick, in press; Resnick, Greeno and Rowland, note 1). These studies reveal the following stages of understanding:

(1) Two-digit numbers can be partitioned into tens and ones, but there is a unique partition. That is, 47 can be shown as 4 tens and 7 ones but in no other way.

(2) Multiple partitions of two-digit quantities are possible, but these must be arrived at empirically by counting. For example, 4 tens and 7 ones (47) can be shown, upon request "with more ones" by removing a ten, and then adding in ones blocks (counting on from 30) until 47 is reached.

(3) Multiple partitions of quantities can be established by trading 10 units for 1 ten, or vice-versa. At this stage, children are confident that quantities are equivalent if trades have been ten-for-one, and they do not spontaneously recount. Our computational models specify this level of understanding as a Trade schema, which develops as a special case of Part/Whole.

CONCLUSION

The cumulating evidence on children's number representation points to a substantial elaboration that takes place during the early school years, at least partly as a direct result of learning about the formal notational system. The Part/Whole schema appears to be central to this development. This schema has been shown to lie at the base of the ability to perform story problems, open-sentence problems, and addition and subtraction with regrouping. It also plays a part in performance on Piagetian tasks--such as class inclusion--that are recognized as important to an operational concept of number. It seems probable that children's mathematical development could be aided by explicit instructional attention to the schema and its applications.

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THE EMERGENCE OF ALGORITHMIC PROBLEM SOLVING BEHAVIOR

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Le **Mathematics Work Group** *du Centre de Recherche et Développement à l'Université de Wisconsin (USA) examine le développement des conceptions et habiletés d'addition et soustraction chez enfants en école élémentaire. Une étude de trois ans, commencé en septembre 1978, a été utilise à assembler information sur les stratégies employées par ces enfants lorsqu'ils résolvent problèmes arithmetiques. Cet article décrit processus algorithmiques qu'élèves emploient à résoudre une variété des problèmes verbals d'addition et soustraction. Une moitié des problèmes exige* **regrouping** *(retenir en addition; ? en soustraction) tandi que l'autre moitié ne l'exige pas.*

Quatre entrevues individuelles ont lieu, la première en janvier 1980 quand les presque 100 sujets etaient dans la classe seconde (écoles américaines, age: 7 1/2 ans). Les autres étaient donnés en mai 1980, septembre 1980 et janvier 1981. Les résultats indiquent l'emergence de stratégies algorithmiques est pareil à l'instruction de la classe en l'usage d'algorithms de calcul. L'utilisation d'algorithms est indépendante de type de problème et operation de mathématiques sauf le problème de Joignant avec un ajoutant (?) absent. (voir Vergnaud, 1976; type ETE_t *, T positif)*

The primary focus of the research program of the Mathematics Work Group of the Wisconsin Research and Development Center is the study of the development of addition and subtraction concepts and skills in young children. The major vehicle for this investigation is a three-year longitudinal study begun in September 1978 with first grade children with an average age of 6 1/2 years. The final data collection point for the study took place in January 1981. A number of variables are under investigation including problem solving behaviors on a specific set of verbal problems, selected cognitive skills, performance on written arithmetic tasks, and the nature of classroom interactions observed in the classrooms of the subjects in question. Details of the study and some earlier results are contained in previous papers presented to the PME (Carpenter, 1980; Carpenter & Moser, 1979; Moser, 1980). In this paper only children's performance on problem solving tasks will be considered. A further restriction is the limitation to problems involving addition and subtraction of two-digit numbers.

BACKGROUND INFORMATION

Subjects. Subjects for the study consist of about 100 children from six classrooms in two elementary schools that all draw from predominantly white middle to upper-middle class neighborhoods. All received instruction from the Developing Mathematical Processes (DMP) program, an activity oriented instructional program developed at the University of Wisconsin. DMP has a strong emphasis on problem solving and during the time period reported here, subjects were instructed in the analysis and solution of verbal problems of the type used in this study.

Data to be reported were taken from four individually administered problem solving interviews that were given in January 1980, May 1980, September 1980, and January 1981. At the time of the first interview, all subjects were in the middle of second grade; thus, by the time of the final interview, all were in the middle of third grade. At the time of the first interview, no formal instruction in the use of computational algorithms had been given. Between the first and second interview, introduction to addition and subtraction without regrouping and addition with regrouping was taught. Summer holidays occurred between the second and third interview. Between the third and fourth interview, the regrouping algorithm for subtraction had been taught.

Problem solving interviews. Each interview included six problem types, two with an additive structure and four with a subtractive structure. Representative problems and the order in which they are given to a child are presented in Table 1.

Each interview consisted of two parts, the first with the six problems containing two-digit numbers for which no regrouping (borrowing or carrying) is required to compute the answer [hereafter described as the "d" problems] and the second part with six problems containing two-digit numbers for which regrouping is required [hereafter described as the "e" problems]. Six different number triples were used for each part. They are listed in Table 2. The assignment of number triples to problem types involved a six-by-six Latin square design resulting in six sets of six problem tasks which were uniformly and randomly distributed across subjects. Problem wording was systematically changed, while retaining the essential semantic structure. The interviews were conducted in a quiet room separated from the child's actual classroom. The child was presented with paper and pencil, and a large set of plastic cubes. Problems were read to the children by the interviewer and repeated as necessary.

Table 1

Representative Addition and Subtraction Verbal Problems

1.	Joining (Addition)	Jacques had 12 pennies. His father gave him 15 more pennies. How many pennies did Jacques have altogether?
2.	Separating (Subtraction)	Marie had 29 candies. She gave 18 of them to Collette. How many candies did Marie have left?
3.	Part-Part-Whole (Subtraction)	There are 31 children in the classroom. Nineteen of them are girls and the rest are boys. How many boys are in the classroom?
4.	Part-Part-Whole (Addition)	Jean-Paul has 17 red marbles. He also has 19 blue marbles. How many marbles does Jean-Paul have altogether?
5.	Comparison (Subtraction)	Chantal has 16 tickets. Her friend Michel has 29 tickets. How many more tickets does Michel have than Chantal?
6.	Joining, missing addend (Subtraction)	Diane has 23 strawberries. How many more strawberries does she have to put with them so she has 37 strawberries altogether?

Table 2

Number Triples Used in Verbal Problems

"d" Problems		"e" Problems	
12,15,17	12,16,28	12,19,31	13,18,31
11,18,29	13,16,29	14,18,32	16,17,33
14,21,35	14,23,37	15,19,34	17,19,36

RESULTS

One of the major questions of interest in this particular set of problem solving tasks was whether subjects would exhibit similar types of solution strategies as they had used with smaller number problems (sums between 5 and 16 and all addends being one-digit numbers). For those problems, a great deal of direct modeling and use of a variety of forward and backward counting techniques had been observed. Or would children resort to algorithmic behavior? A child was coded as using an algorithm if he/she gave direct written or verbal evidence that place-value consideration had been made and that computations were made separately for the ones' and tens' places. We did not record how the actual computation within a particular place was carried out. If, for example, the problem involved the sum $\begin{array}{r}15\\ \underline{+\ 19}\end{array}$, we did not attempt to determine how the child would get the sum of 5 + 9, either by a known or derived fact, or by some counting method. Table 3 presents the results for the four interviews for all six problem types and for both number sizes. Both the percentage of children who used an algorithmic behavior and the percentage of correct answers from among the algorithm users are given.

Table 3

Percentage of Children Using Algorithmic Behavior

	Interview							
	1 (Jan. 80)		2 (May 80)		3 (Sept. 80)		4 (Jan. 81)	
Problem type	d	e	d	e	d	e	d	e
1 Joining (Addition)	24 (20)	25 (21)	61 (58)	69 (53)	67 (56)	60 (45)	90 (88)	92 (86)
2 Separating (Subtraction)	19 (17)	14 (3)	65 (51)	58 (2)	58 (54)	40 (3)	87 (85)	88 (69)
3 Part-Part-Whole (Subtraction)	18 (15)	14 (2)	64 (51)	52 (2)	48 (39)	32 (2)	89 (86)	80 (59)
4 Part-Part-Whole (Addition)	32 (31)	24 (19)	70 (65)	72 (60)	66 (62)	61 (44)	92 (87)	95 (85)
5 Comparison (Subtraction)	16 (14)	14 (2)	50 (38)	45 (3)	41 (35)	27 (3)	78 (73)	81 (65)
6 Joining, missing addend (Subtraction)	18 (17)	10 (2)	39 (27)	35 (3)	28 (23)	26 (3)	59 (54)	54 (40)
Actual number of subjects	96		96		93		93	

(Numbers in parentheses represent percentage of total subjects who used algorithmic behavior who also solved the problem correctly.)

The immediate impression is that the increase in frequency and correctness of use of algorithmic behavior mirrors instruction in computational algorithms. Paper-and-pencil arithmetic skills tests administered independently of the problem solving interviews give exactly the same results in terms of ability to use a computational algorithm correctly. The great majority of errors made with the regrouping subtraction algorithm in the early stages prior to formal instruction with that algorithm were of the type well know to teachers, which is exemplified by $\begin{array}{r}31\\-18\\\hline 27\end{array}$, where in the ones' place the child follows the rule of "subtract the smaller from the larger" without any regard to the meaning of the entire number.

Of more interest than simple correctness is the different pattern of use of algorithmic thinking for problem 6, the Joining, missing addend task. A reasonable explanation for the much lower incidence of algorithmic solution is the semantic structure of the problem. Using the specific example given earlier, the wording strongly suggests that the best literal translation of that problem is the number sentence $23 + \square = 37$. However, of those children who elected to use symbolic representations almost all chose to use the vertical computational form rather than horizontal sentences. The vertical counterpart to the sentence written above is an awkward one, totally unfamiliar to children who had only seen the traditional form. It would appear that the children who realized this fact decided to not proceed in an algorithmic fashion, even though their behavior on other subtraction problems indicated that they could correctly use the subtraction algorithm. The most frequently used alternative strategy for this sixth task was Counting Up.

Another facet of the study was to investigate the relationship between the use of written symbolic representations and the use of algorithmic solution processes. If a sentence, either horizontal or vertical, was written, it was almost always the case that the sentence was written before the solution process was initiated. This is contrary to the results of an earlier pilot study (Carpenter, Moser, & Hiebert, 1981) where, when smaller numbers were involved, the children wrote the sentence after solving. In this latter study, however, the experimenter directed the child to write a sentence. In the present study, using written symbolism was at the discretion of the child. There was a very large number of children who did not write sentences, but still solved algorithmically. This was especially true with the "d" problems. In fact, at the time of the fourth interview, almost half of the subjects did not write a sentence for the addition problems without regrouping. The success rate for algorithmic students who did not write sentences was very high, due to the fact that these were probably the brighter students who are likely to solve correctly anyway.

While much of the discussion has dealt with the use of algorithmic solving, it is appropriate to briefly characterize the behavior of those children who did not use algorithms. The results are essentially similar to those we have gathered using the same subjects, but with the smaller sized number problems. Problem structure appears to be the most powerful factor in determining the choice of strategy. Subtractive strategies seem to predominate for the Separating problem while additive strategies are most evident for the Joining, missing addend problem. Again, the only place where Matching appears is with the Comparison problem. As noted earlier, problem 6, the Joining, missing addend was the task that had the least number of algorithmic solutions. As a result, it was also the problem with the greatest frequency of so-called "Heuristic" (Carpenter, 1980) strategies. I take this as further evidence that children are capable of inventive behavior (Moser, 1980).

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MAITRISE ET DISPONIBILITE DU NOMBRE CHEZ L'ENFANT DE 7-8 ANS

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The enquiry and research on the children's behaviour in the "cycle préparatoire"(1) began in 1979 and was conducted by the Mathematic research unit (primary level) of the Institut National de Recherche Pédagogique (I.N.R.P.). The results are now being analysed.

The general topic of this article is to present, on the one hand, the scope of the inquiry, on the other hand, the results connected to "number".

This inquiry was carried out in 66 classes covering a total of 1459 pupils. Theses classes were selected in order to constitute a representative sample. 990 children were chosen for an individual testing session to ensure a survey of their solving problems strategies. The whole sample were given written exercices.

In the exercices concerning the counting of sets and afterwards of numbers we could expect a better achievement when the exercice consists of small numbers and a weaker achievement when it comprises large ones.

The tendency is globally confirmed but the fluctuations in the results lead to take into account variables specific to each situation which characterize the real complexity of the task. Specially, it is necessary to set apart the exercices or items in which children are explicitly asked to count from the others.

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(1) "Cycle préparatoire": 1st year in a primary school (The year they learn how to read, to count, etc.).

L'enquête "Evaluation des comportements des élèves au Cycle Préparatoire" mise en place en 1979 par l'Unité de Recherche mathématique Elémentaire de l'I.N.R.P. est actuellement en cours d'analyse. L'objet de cette communication est de présenter dans le cadre de l'enquête,quelques résultats relatifs au nombre.

PRESENTATION DE L'ENQUETE

Cette enquête a été réalisée auprès de 1459 élèves répartis dans 66 classes , choisis de façon àconstituer un échantillon représentatif.L'objet de l'enquête était d'évaluer non seulement quelles connaissances mathématiques maitrisent les élèves, mais encore quand et comment elles sont disponibles.Celà nous a conduit à utiliser deux types d'épreuves:

-pour tout l'échantillon, des épreuves collectives proches de situations scolaires, évaluables en termes de réussite ou d'échec.

-pour 990 enfants,des passations individuelles permettant une description des actions, une observation des "stratégies", un recueil des justifications verbales des actions. Dans ce cas,nous avons retenu des situations éloignées des situations scolaires, pour voir comment des savoirs ou des savoir-faire sont mobilisés dans des situations où l'enseignant n'a pas appris à l'élève à les utiliser.

Description des épreuves individuelles:

D'une part, ellesenvisagent le nombre sous ces deux aspects:

-aspect cardinal, étudié dans des situations où le dénombrement est tantôt demandé explicitement, tantôt une procédure efficace mais non explicitement demandée.

-aspect ordinal,dans des situations sollicitant des activités de comparaison de nombres (mise en ordre, recherche du prédécesseur et du successeur).

D'autre part,dans les deux cas nous avons cherché à évaluer les problèmes posés par la numération orale (connaissance et usage de la comptine) et à comparerles performances en numération orale et en numération écrite.

Epreuve 1 : LES JETONS

Un tas de 47 jetons est présenté à l'enfant; on lui demande de les compter, de dire puis d'écrire le résultat; mêmes consignes avec un deuxième tas de 8 jetons; On demande combien il y a de jetons en tout.

Epreuve 2 : LE QUADRILLAGE

On présente un quadrillage 6x6 dont certaines cases (21) sont remplies par un jeton; d'autres, non. On demande à l'enfant d'aller chercher"juste ce qu'il faut de jetons" (15) pour complèter le quadrillage.

Epreuve 3 : LES ALLUMETTES

On présente deux séries d'allumettes,l'une où 8 allumettes font une ligne brisée,l'autre disposée sous la première, où7 allumettes forment une file plus longue; on demande "Y a-t-il plus d'allumettes là ,ou plus d'allumettes ici, ou autant d'allumettes là que là ?". On demande à l'enfant de justifier sa réponse.

Epreuve 4 : LE LIVRE

Les pages 41, 69, 92 et 125 ont été ôtées d'un livre. On demande à l'enfant 1° de les ranger en ordre sur la table

2° de les remettre à leur place dans le livre

3° de trouver pour chaque page quel est le "numéro" de la page précédente

4° d'écrire ce numéro

PRESENTATION DE QUELQUES RESULTATS

Epreuve 1: Les jetons

-dénombrement: la procédure employée par presque tous les enfants (environ 90%) est de déplacer les jetons un à un en récitant la comptine.

Résultats

(n=495)

Dénombrement	tas de 47	tas de 8
correct	43	93
erreur de 1	16	3
autres cas	41	4

-nombre oral, nombre écrit: pour le tas de 47 jetons, écrivent correctement le nombre annoncé oralement ;

-97% de ceux qui ont dénombré correctement

-94% de ceux qui ont fait une erreur d'une unité

-50% de ceux qui ont fait une erreur de plus de 3 unités

-addition:procédure employée et correction de la réponse

	correct à1 près	incorrect
calcul mental explicite	6	2
continue la comptine	22	8
recompte tous les jetons	20	22
pas de dénombrement apparent	5	12
autres cas	-	3
Total (n=495)	53%	47%

Epreuve 2: Le Quadrillage , procédure employée et performance

	correct à 1 près	incorrect
prend un seul jeton	-	12
pas de comptage explicite	13	26
compte explicitement cases ou jetons	16	9
compte explicitement cases et jetons	20	4
Total (n=495)	49%	51%

Epreuve 3: Les Allumettes,procédure employée et performance

	correct	incorrect
a compté ou dit avoir compté	37	9
évoque le dépassement	-	29
autres cas	5	20
Total (n=495)	42%	58%

Epreuve 4: Le Livre

-liaison entre les deux premières parties de l'épreuve

rangement des 4pages sur la table / placement correct dans le livre	correct	incorrect
0 page	4	30
1 ou 2 pages	5	10
3 pages	6	10
les 4 pages	24	11
Total (n=495)	39%	61%

-liaison entre la troisième et la quatrième partie de l'épreuve

Nous présentons pour la seule page 69 (bien ou mal placée),les
résultats àla consigne:" Dis quel est le nombre qui vient avant"
" Ecris le nombre qui vient avant"

	page bien placée	mal placée
oral et écrit corrects	69	33
oral incorrect, écrit correct	24	28
oral correct, écrit incorrect	-	-
Tout incorrect ou rien	7	39
Total	100 (n=263)	100(n=232)

Quelques remarques suggérées par ces résultats

Dans les épreuves citées ci-dessus,relatives pour les unes au dénombrement de collections et pour les autres à la suite des nombres,on pourrait s'attendre à ce que la taille des nombres mis en jeu soit déterminante pour la réussite.Cette tendance est globalement vérifiée mais les fluctuations observées contraingnent à formuler d'autres hypothèses: en dehors aux variables liées aux onnaissances mathématiques, des variables spécifiques à chaque sitution interviennent pour déterminer la complexité de la tâche et modifier les résultats attendus.

Ainsi à l'épreuve quadrillage, où il suffisait de compter 7 et 8 (ou 15) pour réussir seulement 34% des enfants réussissent alors que 93% d'entre eux savent dénombrer un tas de 8 jetons et 43% un tas de 47 jetons. Ainsi la disponibilité de la comptine,même très bien maitrisée par les enfants, reste très dépendante de la situation dans laquelle elle est utile; si, outre le dénombrement l'enfant doit également accomplir une autre tâche (manipulation, déplacement) ceci complique le problème au point de rendre le comp comptage incorrect ou indisponible pour une grande partie d'entre eux;le fait, par ailleurs que le comptage ne soit pas explicitement sollicité, entraine les enfants à recourir à d'autres procédures,la plupart du temps inefficaces. Ainsi,l'échec relatif des enfants dans des situations simples mais éloignées des situations familières de l'apprentissage, nous amène à souligner la distinction qu'il convient de faire entre maitrise et disponibilité du Nombre et de la numération.

Enfin, il nous parait nécessaire de souligner également les difficultés spécifiques liées à la numération orale;cette difficulté est généralement occultée par les épreuves du type "papier-crayon" ou dans des épreuves seulement orales. D'après l'enquête, (épreuve du Livre en particulier) le travail sur la numération écrite pose aux enfants moins de difficultés que celui sur la numération orale.

Subtracting fractions with different denominators

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Résumé

Cet article est un résumé de recherche développante telle qu'elle a été fait dans le cadre du développement de l'enseignement mathématique de IOWO. Au cours de cette recherche on a construit un cours partiel où les élèves suivaient un chemin d'algorithmisation graduelle qui devait les conduire sûrement mais sans les presser à l'algorithme de la soustraction (et éventuellement aussi à celui de l'addition) des fractions à dénominateurs différents. C'est une question centrale si le principe de la schématisation graduelle ou progressive s'applique aussi à l'algorithmisation graduelle (voir Freudenthal, 1981). Cela se montrerait dans le comportement de solution des élèves participant à cette recherche:

- progression de la schématisation, et
- l'application de raccourcis,

parmi d'autres au moyen de propriétés des transformations laissant invariantes les raisons des magnitudes.

1 *Preliminary instruction*

The research to be reported regards the final step in mathematising the comparison of distribution situations. It strongly depends on the preliminary instruction on fractions which the students had received. A paper containing a sketch of this instruction can be obtained from the author's office (OW & OC, Tiberdreef 4, 3561 GG Utrecht, The Netherlands), or personally at the meeting.

2 *The developing research - some results*

2.1 *The subjects*

The subjects were all pupils of a fourth grade (20 pupils) and a fifth grade (24 pupils), both of them classes of a so-called stimulation school, that is, a school getting an extra teacher on the ground of a social indication (underprivileged pupils). We chose for this kind of schools since it may be expected that the difficulties into which many primary school pupils traditionally get with fractions, will show up most clearly with this population. In the description of the results we will pay the main attention to fifth graders because the group fourth graders dropped out rather early. The main reason was insufficient acquaintance with ratio tables when proportion problems had to be solved.

2.1 *First start*

Since one of the objectives of the investigation was retention of insight in the course of the designed instruction, we started with an accelerated run through some of the learning processes of the preliminary instruction. These were:

Distribution situations

Exploring distribution situations and performing distributions with a view on the twin meaning of fraction, which both aims at the distribution situation and at its result - contriving new distribution situations of the resul is given and storage of the numerical data in a table. Example: everybody gets $\frac{3}{4}$ of a pancake. Distribution situations:

Pancakes	3	6	9	12						
Children	4	8	12	16						

It is secretly assumed that the table is 'finished' as soon as it contains about ten situations. Such open tasks which elicit diverging solutions however, provoke deviating from this convention. The algorithmic activity invites continuation, even beyond the task set. It is an advantage of this approach that it does justice to the childrens' inclination to algorithmising and their need of building personal algorithms (cp. Carpenter, 198 Hart, 1981). It is or can be important disadvantage that the intended algorithmic rote can gravely be disturbed by noticing of, and yielding to, neighboring though irrelevent regularities. This happened particularly in the beginning of the investigation. Example:

Pancakes	3	6	9	12	16	20	26
Children	4	8	12	16	20	26	32
				↑	↑	↑	

We do not report the results of this activity because it was only a repetition

Fitting a 'new' situation

Example: 16 pancakes are ordered for 24 children. Does this situation fit into the previous one? The attention of almost all pupils is directed on the fitting into the previously constructed table:

Pancakes	3	6	9	12	15	18		
Children	4	8	12	16	20	24		

The situation that is to be fitted in is immediatly compared with:

P		18
C		24

In some cases the table was extended to facilitate the comparison of:

x2

P		9		18
C		12		24

x2

with

P		16
C		24

One girl only hit on the idea of reasoning backwards from the new situation with finally the remark that she had stopped because 1½ child was impossible:

←

Pancakes	1	2	4	8	16	
Children	$1\frac{1}{2}$	3	6	12	24	

←

As the teacher asked what was the the share of one child, she arrived at:

P	$\frac{2}{3}$	
C	1	

with the conclusion that this was less than $\frac{3}{4}$ of a pancake.

Contriving 'intermediate values'

For 'each child gets $\frac{3}{4}$ of a pancake' the following table was made:

Pancakes	$\frac{3}{4}$	$1\frac{1}{2}$	$2\frac{1}{4}$	3			6				9	
Children	1	2	3	4	5	6	7	8	9	10	11	12

The pupils of the fourth grade now lag considerably behind these of the fifth. Though most of them know to deal with things, there are serious shortcomings. The most striking is that now 14 fourthgraders among 20 fall back to drawing diagrammes to recall the distribution situations as a support in filling out the table. Better than half of the fourthgraders were satisfied with a few distributions:

Pancakes	$\frac{3}{4}$	$1\frac{1}{2}$	$2\frac{1}{4}$	3		$4\frac{1}{2}$		6		
Children	1	2	3	4	5	6	7	8		

Almost alle fifthgraders got the following solution:

Pancakes	$\frac{3}{4}$	$1\frac{1}{2}$	$2\frac{1}{4}$	3	$3\frac{3}{4}$	$4\frac{1}{2}$	$5\frac{1}{4}$	6	$6\frac{3}{4}$	$7\frac{1}{2}$	$8\frac{1}{4}$	9	$9\frac{3}{4}$	$10\frac{1}{2}$	$11\frac{1}{4}$	12
Children	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16

Two pupils made an even more extended table, while three were satisfied with a more restricted one.

Interim balance

This justified the conclusion that the fourthgraders fell short in carrying out the tasks set because they lacked experience in working with ratio tabl and in algorithmising ratios. A decisive part is played by ratio conserving mappings of domains of magnitudes (Kirsch, 1969; Freudenthal, 1978), whose properties, though partly spontaneously applied by the pupils had insufficiently been made conscious in previous learning processes. It was then decided to continue the investigation with te fifth graders only.

2.3 *Continuation of the investigation with the fifth graders*

Situations to be fitted

When distribution situations were compared, the immediately comparable ones caught the eye. The pupils discovered that comparing (and making up the difference) was easy in the case of an equal number of children. This fact was verbalised. Their attention was shifted to the more tedious cases. When $\frac{4}{6}$ and $\frac{3}{5}$ were to be compared, the teacher suggested they were equal. When they attempted to prove this wrong, a few pupils proceeded by transforming the situation into an equivalent one:

P	4	3	2
C	6	$4\frac{1}{2}$	3

while everybody grasped that:

P	3	
C	$4\frac{1}{2}$	

was absurd as a distribution situation. The task set to compare the situations, however was now simplified (an equal number of pancakes) which justifies the conclusion that $\frac{4}{6}$ is more than $\frac{3}{5}$. Only the difference could not yet be determined. As a matter of fact many pupils easily picked up th idea of transforming the situation in order to simplify the comparison. So of such comparison tasks were processed on paper. The observation team (three people) noticed that a small number of pupils worked rather systematically towards easily comparable situations. With a view on the sequel one of them was given the opportunity to explain his method: 'I simply cou through until the number of children is equal', he laconically commented.

2	4	6	8	10
3	6	9	12	15

He did not continue the table but when asked why he did not, he answered: '15 is also met in the other table':

$\frac{4}{5}$:	4	8	12
	5	10	15

From the written material it afterwards appeared that about 60% of the pupils had worked in a more or less similar way, either spontaneously or by imitation. What strikes in this solution is the applied shortcut. There is no unconcerned production of distribution situations followed by comparing. On the contrary. There is a permanent reflexion on the production of distribution situations, there are intermediate checks and at the first possible opportunity the production is stopped, the strategy is changed to work from the new situation toward the desired one. If there were divergences they regarded:

- no determination of the difference (two pupils);
- restriction to qualitative statements on order (three pupils);
- no application of the shortcut, which means producing more distribution situations than strictly needed (three pupils);
- errors in determining the difference (two pupils).

Sequel

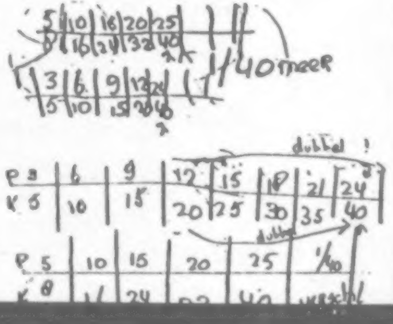
In the sequel we pay attention mainly to the progression in the process of algorithmisation by means of the table method. Besides distribution situations other applications were processed. By a permanent switch to new applications the systematic approach, the retention of which is aimed at, is detached from its band with reality and abstracted to become the final generally applicable algorithm for subtracting fractions with different denominators. We give the results of two examples, which were dealt with as late as two weeks after the previous experiences.

Example 1: comparing $\frac{3}{5}$ and $\frac{5}{8}$ on the basis of real distribution situations. More than half of the 24 pupils has become familiar with the systematics of the approach. A few pupils are eager to streamline their methods, as appears from the reports below and is witnessed by the applied shortcuts.

Examples of work

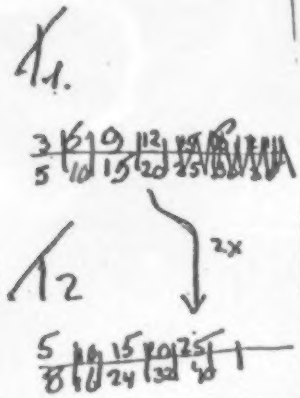
Sylvia's work:

Sylvia discovered afterwards that in the table of $\frac{3}{5}$ one could even skip three terms. So she made new tables, starting with that for $\frac{5}{8}$ and applying the shortcut in that of $\frac{3}{5}$.


5 10 16 20 25
8 16 24 32 40
3 6 9 12 24
5 10 15 20 40
1/40 meer
dubbel !
P 3 6 9 12 15 18 21 24
K 5 10 15 20 25 30 35 40
dubbel
P 5 10 15 20 25 1/40
K 8 16 24 32 40

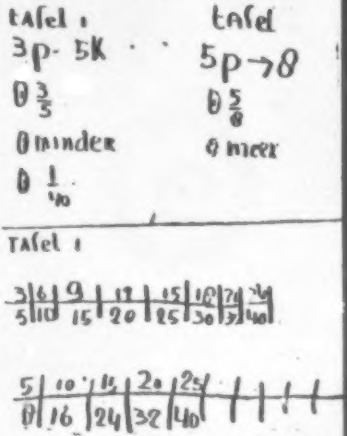
Some other pupils had the same solution with the difference that the redundant situations are struck out:

Jacqueline's work:



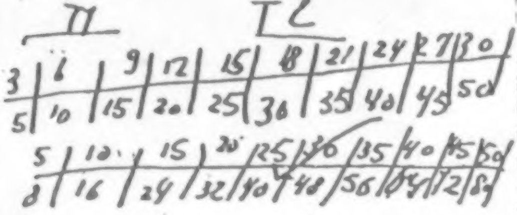
The same shortcut, though Jacqueline cancels an extra term and uses the same mapping property as Sylvia

Peter-Jan's work:



This work is illustrative for the group in general. Most of the pupils proceed this way, that is automatically redundant solutions

Bert's work:



Extending both tables beyond the need, followe by mutual comparison was not unusual

Example 2: comparing $\frac{3}{4}$ and $\frac{4}{5}$ in an application (mixtures), (cp. Noelting, 1980, Noelting and Gagné, 1980). Problem: somebody makes coffee with a machine. One time: three spoons of coffee for four cups. Another time: fou spoons for five cups. Which coffee is stronger and what is the difference? With a view on the shortcut and the progression in schematising the solutions found by the pupils can be characterised as follows.

Observed solutions

1 Making tables for both situations and stopping at the first that can easily be compared to the other:

Spoons	3	6	9	12	15
Cups	4	8	12	16	20

and

Spoons	4	8	12	16	
Cups	5	10	15	20	

Seven pupils did it this way.

2 The same but by comparing the situations (three pupils):

S	12	
C	16	

and

S	12	
C	15	

3 First making tables of a size that is judged sufficient and then passing to comparison and determination of the difference. Among the six pupils who proceeded this way there were two who worked by continued doubling, which just excludes the appropriate situations:

Spoons	3	6	12	24	48
Cups	4	8	16	32	64

and

Spoons	4	8	12	16	
Cups	5	10	15	20	

One pupil filled out the tables up to:

S	32	
C	40	

and

S	30	
C	40	

respectively

and arrived at a strength difference of $\frac{2}{40}$ spoon per cup.

4 Constructing tables for both ratios up to the first easily comparable situations (thus as sub 1) but with shortcuts under way. There were six pupils who approached it this way. Four of them did so only one table, namely:

Spoons	4	8	16
Cups	5	10	20

while the two remaining pupils did it in both tables:

Spoons	3	6	12	15
Cups	4	8	16	20

and

Spoons	4	8	16
Cups	5	10	20

It appears that pupils of this group gradually come to grips with the systematics that will eventually be fixed in the algorithm of subtracting (and adding) of fractions with different denominators.

3 *Conclusion*

Algorithmisation was interpreted in the sketched developing research as a process that delays fixing the final algorithm. In the present case, subtracting of fractions with different denominators the process was characterised by:

- progression in schematising;
- performing shortcuts.

The progression in schematising expressed itself above all in the way the tables were simplified by the pupils and adapted to the needs of the chose solving path. The aspect of performing shortcuts is sufficiently illustrat by our examples. The results achieved are always considered in relation to the partial course in state of development. It was the objective to investigate how a learning process of a larger group could take place within a margin of differentiation. By the last examples we gave, the process of algorithmisation is not yet completed. The investigation is open ended. It should be stressed that the investigation took place in a real instruction situation. Mathematics is being considered as a human activity. So childre need to build their own algorithms can be met. Learning algorithms takes place gradually rather than algorithmically.

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9

LE CERCLE

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The Circle

We have chosen the concept of the circle for children from 8 to 11 years old. In the course of an experiment we proposed the widest possible range of problems and situations then we proceeded to the theoretical study of the concept of the circle in order to determine a number of definition which gave us the opportunity to study the procedures used by the children.

We then built up a didactical sequence consisting of six different situations. The purpose of this sequence was designed to induce the emergence of concept of the circle. Every situation is analysed "a priori" the analysis is then compared to the children's procedures.
The results of these comparisone form basis of studies from which we draw certain conclusions. After the sequence, each child is interviewed individually.

We then compare the test results with the expected knowledge acquired in the course of the didactical experiment. The didactical sequence was carried in two classes simultaneously. This enables the comparison between the attitude and procedure of each teacher in similar situations. Finally we studies with one of the classes the language used by the children and the teacher and we tried to draw conclusions concerning strategy of the teacher.

ORIGINES DE LA RECHERCHE

La géométrie dans l'enseignement élémentaire a un domaine assez restreint ; elle se borne à deux activités :

- l'étude des formes géométriques simples (carré, rectangle, cercle etc...)
- l'étude de transformations (symétries, rotations, etc....)

Nous avons choisi de nous intéresser à l'étude des formes géométriques. Dans presque tous les manuels d'enseignement, ces études sont des leçons de vocabulaire, dans lesquelles les objets géométriques sont nommés et montrés. Nous avons pensé qu'il était possible d'aller plus loin qu'une simple monstration ; nous avons essayé de le faire à propos du cercle. Ce n'est pas au hasard que nous avons choisi le cercle, mais parce que c'est la figure qui donne la plus grande illusion de simplicité :

- elle est perceptivement reconnue très tôt.
- elle est facilement tracée. En effet, le compas est d'un usage très aisé.

PRE-EXPERIMENTATION

Avant de nous lancer dans la fabrication d'une séquence didactique à propos du cercle, nous avons essayé de recenser les différentes conceptions du cercle que les enfants de l'école élémentaire sont capables de mettre en oeuvre. Pour cela, nous avons fourni à des enfants de 8 à 11 ans des situations problèmes très variées : reconnaissance de formes, messages décrivant des formes géométriques, trajectoires circulaires (coins de porte, pendule), positions relatives de disques, partages de disques en secteurs isométriques, homothétie de cercles etc....
Nous avons noté et étudié les différentes procédures utilisées par les enfants. Il nous a paru alors indispensable d'essayer de recenser théoriquement diverses conceptions possibles du cercle. Cela nous a aidé à rattacher avec une plus grande certitude telle procédure à telle ou telle conception du cercle. Pour cela nous n'avons pas fait une recherche exhaustive, mais nous avons trouvé des définitions qui permettent de décrire assez bien les conceptions des enfants.

EXPERIMENTATION

Nous avons constitué une séquence didactique formée de six situations. Chacune de ces situations est construite de façon à favoriser l'émergence de telle ou telle des conceptions répertoriées lors de la pré-expérimentation. De plus, les difficultés technologiques liées à la situation sont un des facteurs qui nous permettent de privilégier l'émergence d'une conception donnée. Ces situations sont présentées aux enfants ; on étudie leurs procédures et on les compare à l'analyse qui en a été faite à priori. En particulier, nous décrivons en détail une situation pour laquelle il ne s'est pas passé exactement ce qui était prevu par l'analyse à priori, et nous tentons d'en donner une explication.

Chaque situation a été proposée dans deux classes de CE_2 (enfants de 8 - 9 ans) ; elles ont été expliquées de la même façon aux deux maîtresses qui avaient les mêmes contraintes :

- les situations devaient être présentées exactement comme nous les avions crées.
- chaque séance devait débuter par un rappel des séances précédentes et devait se terminer par une phase collective de synthèse. Pour le reste, choix du vocabulaire, déroulement de la classe, prises de décision, choix des notions à renforcer etc...., elles étaient libres de s'organiser à leur gré.

. Trois situations sont destinées à favoriser plus ou moins l'utilisation de la constance de la courbure, cela en concurrence avec d'autres conceptions (utilisation de l'invariance du rayon, par exemple). Il s'agit de :

- reconstitution de disques découpés le long de rayons - construction d'un secteur angulaire manquant à l'un des disques.
- reconstitution de couronnes découpées le long de rayons - construction d'un secteur angulaire manquant à l'une des couronnes.
- reconstitution de cercles à l'aide de quatorze arcs de cercles provenant de quatre cercles différents.

. Les enfants doivent construire le centre d'un cercle donné.
. Les enfants doivent prévoir la trajectoire du coin d'une porte.
. Les enfants doivent dessiner un cercle et envoyer un message téléphonique pour qu'un autre puisse dessiner un cercle de même taille.

L'analyse est faite situation par situation. Nous étudions les conduites et les procédures des enfants et nous les comparons à nos prévisions. Dans la situation de la trajectoire du coin de la porte, les réactions des enfants diffèrent assez sensiblement de ce qui était prévu. D'une part nous en analysons les raisons, d'autre part nous étudions les prises de décision de chacune des maîtresses placées devant un imprévu. Nous avons essayé de cerner l'effet d'apprentissage de la séquence didactique. Pour cela nous avons construit six tests que chaque enfant a passé en entretien individuel. Nous avons dépouillé ces tests et comparé les résultats avec les performances des enfants lors de la séquence. Nous avons analysé soigneusement les phases de rappel au début de chaque séance. En effet, il nous a semblé, qu'il était possible de trouver dans ce discours, des indices qui nous permettent de cerner les notions que la maîtresse a l'intention de faire retenir aux enfants. Nous avons essayé de rapprocher ces notions des notions bien assimilées c'est-à-dire ayant donné des résultats positifs aux tests. Nous avons aussi le matériel qui nous permettrait d'étudier le rôle particulier joué par certains enfants lors du déroulement de la séquence, mais nous n'avons pas encore réalisé cette étude.

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KNOWLEDGE, ABILITIES, AND PERFORMANCE IN LEARNING SUBJECT MATTER FROM ARITHMETIC AND GEOMETRY.

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Les processus d'apprendre deux matières de l'arithmétique et de la géométrie à la 5e classe sont analysés par une équipe de récherche à l'université d'Osnabrück. On peut supposer que le succès en apprendre une matière mathématique est au moins dépendant (1) des connaissances acquises auparavant, (2) des aptitudes intellectuelles et mathématiques, (3) de l'exécution actuelle d'un problème mathématique. Les structures des deux matières et les natures des connaissances présumées aux élèves de la 5e classe sont très différentes (entre les matières). Les dimensions d'aptitude obtenues psychométriquement décrivent les conditions individuelles des processus d'apprendre seulement en gros. C'est pourquoi il est proposé d'analyser des processus cognitifs minutieusement à la base des théories sur le résoudre des problèmes. Cette analyse peut produire des connaissances sur le développement différentiel des concepts mathématiques aux élèves. Un plan d'une récherche empirique est présenté brèvement.

As part of a research program about teaching and learning (supported by Deutsche Forschungsgemeinschaft) a research group at the University of Osnabrück investigates learning processes of 5th grade pupils in mathematics' instruction. Two different topics from arithmetic and geometry (place value systems, axial symmetry) have been selected. We hope to get some insight into children's mathematical thinking and concept formation in these main fields of 5th grade mathematics. Common aspects and differences in learning the mentioned topics are to be investigated to get some clues improving mathematics' instruction.

Only a brief overview of the present phase of the study can be given here as its theoretical foundation and empirical design are still developed.

Pupils' success or failure in mastering mathematical subject matter can be determined (if you let motivational and classroom interaction aspects aside) by at least three factors: (1) their previously acquired knowledge (concerning the matter), (2) their

general and specific intellectual/mathematical abilities,
(3) their actual performance of a task. These aspects are, of course, interdependent and they are accessible to empirical research to different extents. Strictly speaking, only the *results* of actual performances can be recorded, and it is a matter of theoretical considerations and empirical design to enable conclusions about the three mentioned aspects and their interrelations.

Some questions concerning these aspects are handled by our present investigation more or less intensively:
(1) How are these different topics represented cognitively (what do the pupils know about them) before the teaching-learning process in our investigation starts?
(2) Are there specific abilities which enable pupils to master one of the subjects with more success than the other and, if yes, how can we describe or even explain the abilities?
(3) Which knowledge about possible and necessary steps of successful performance can we get from an intensive investigation of actual performances?
(4) Which conclusions about the acquisition and change of concepts can be drawn from the whole investigation?

Let us first consider knowledge pupils have acquired about place value systems and axial symmetry up to fifth grade. Some children heard about non-decimal systems in primary school (up to fourth grade in the FRG) and some did not. But this does not seem to make much difference when they write tests on the topic in the fifth grade: Most pupils in previous stages of our study did not know a considerable amount of the matter. - Of course, they are all drilled in handling one special place value system: the decimal system - but they don't know its structure. The decimal system is *the* single medium of counting and computing for the pupils. They are no longer aware of the specific features of the numerals they manipulate. It may be possible to interpret this in terms of a theory about "the representation of knowledge in memory" (cf. Rumelhart and Ortony, 1977) which has gained some importance in the last few years. The fifth-graders might have a well established schema for counting and computing in which the decimal system is a constant. Instruction in non-decimal systems has the task of generalizing the schema by breaking open this constant, in detail: by replacing the fixed base and place values with variables.

Another aspect of this subject matter must be mentioned which constitutes the most important difference between arithmetic and geometry: it is the kind of concepts pupils deal with (cf. Schmidt in this volume). Number is a very abstract concept representing the power of a set. It is laid down in simple symbols by convention.

Contrary to the concept "number", the concept "axial symmetry" can be modeled in a way that you can directly recognize its features. You can, at least with help, detect the rules of axisymmetric mapping on an axisymmetric figure.

Axial symmetry is introduced in the primary school on a propaedeutic level but the rules of constructing exactly axisymmetric maps are not taught until the fifth grade. So the prerequisites of the learning processes, as far as subject matter is concerned, could hardly be more different.

The question about specific (mathematical) abilities (2) is a rather delicate one because it leads to a theoretical and methodological matter in dispute. Assessment of abilities has been a matter of psychometricians for decades. They constructed intelligence tests and other aptitude or achievement tests for diagnostic and especially prognostic purposes. By factor analyses, dimensions were found (among others) which could be interpreted as representatives of mathematical abilities, for instance: "number", "space" ("visualization", "spatial relations") (cf. Treumann, 1974).

This means roughly: people differ notably in responding to groups of tasks which were considered as indicators for the different dimensions. So statistical attainment of the mentioned dimensions "number" and "space" seems to indicate that "there are" or "can be" different aptitudes for learning arithmetic and geometry. But such a statement is superficial because it does not explain how these aptitudes come into existence, i. e. which cognitive mechanisms are "responsible" for them.

In the last decade, "cognitive psychologists" (and "cognitive scientists") have taken initiatives for redefining intelligence in the light of cognitive theories, especially those of human information processing respectively problem solving (cf. Resnick, 1976a).

This new view leads us to the third question. If the psychometrically defined abilities are too superficial to explain differential success in mastering mathematical tasks, perhaps minute registration and analysis of actual performances of such tasks may reveal some of the cognitive processes responsible for success and failure.

Theories of problem solving (e. g. Klix 1976[3], p. 637 ff) may serve as a frame for such an analysis because performing tasks at school is problem solving in many cases, especially when new subject matter is introduced.

Here is a very much condensed description of what is a problem. There is an initial situation, or better its cognitive representation, with its specific features and their interrelations. This initial state of a situation is to be changed into a different, final state, the goal, with its specific features and their interrelations. There are operations to transform the initial state into the goal. If these constituents (initial state, goal, transformations) are all well defined there is no problem. If one or two of the constituents are not or ill defined and if the individual is aware of this and wants to change it, there is a problem for him or her. You can describe different types of problems according as which of the constituents are ill (not) defined. Different strategies of problem solving are required for different types of problems.

Here is not the place to go into the details but it is obvious, that the internal representation of mathematical tasks as problems depends on the student's previously acquired knowledge about features and relations of the task, about possible transformations and their combination. So, it is necessary to assess students' relevant knowledge, and, for purposes of research, it is suitable to approximate the different individual states of knowledge as far as possible. Thus, when presenting a task to the students, you know which type of problem they will have to solve and that it is the same type for all of them.

As a tool for analysing mathematical tasks as types of problems and for analysing the process of problem solving, instructional task analysis may turn out to be fertile. In an every-day definition you could say, that task analyses "translate 'subject-matter' descriptions into psychological descriptions of behavior"

(Resnick, 1976b, p. 51). This translation can be done in advance to produce the structure of an idealized performance and it should be compared with the empirically assessed performances. The results of these analyses may serve to answer questions (3) and (4). As far as task analysis elucidates "the relations of activity *during* learning and competence that *results from* learning" (Resnick 1976b, p. 53) it may yield some clues for answering question (2).

Finally, a very short sketch of the design of our investigation is given to indicate how empirical answers to the questions might be attained.

At first, the fifth-graders will write pre-tests about the topics place value systems and axial symmetry. Thus, their previously acquired knowledge will be assessed.

Then two divisions of pupils will be instructed beginning with either of the topics alternatively. Groups of ca. 8 pupils will be arranged in order to get favourable conditions for instruction. The pupils will be taught for some lessons. Then criterion-oriented tests will be written which might be an indicator for the pupils' competence for mastering the subject matter. Afterwards the topics will be exchanged between the two divisions of pupils and teaching the new topic will start in either division.

The central part of either curricular unit is programmed in a teaching machine (see Schmidt , 1980). These parts deal with the basic structure of place value systems and the rules of axisymmetric mapping.

After the instruction in groups, single pupils will work at the programmed teaching machine with a (human) teacher sitting by the side of the pupil, supervising his performance of tasks, recording his thinking aloud and giving small hints where necessary. Thus, we shall gain data which will serve as material for an intensive analysis of problem solving and concept formation processes.

Note that we shall be able to assess which types of problems are concerned when we shall have evaluated the results of the pre-tests and the criterion-oriented tests. We also hope to approximate the pupils' states of knowledge by the instruction in groups, so that there will not be many inter-individual differences concerning the types of problems.

A post-test will conclude the whole investigation.

Changes or stability in knowledge and possible discrepancies with regard to both topics must be analysed. Thus, we may get some hints at specific aptitudes. Analyses of both knowledge and process data might enable us to develop didactic proposals for improving instruction.

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ABOUT THE DIAGNOSIS OF SUBJECTIVE FORMATION OF CONCEPTS OF CHILDREN (ESPECIALLY THE CONCEPT "AXIAL-SYMMETRY")

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For the development of "analysis-instruments", recording the learning process, it is fundamental to understand the investigator's conception of this learning process.
The following text describes this conception. The conditions for the construction of a "diagnosis-course" will be derived. The structure of such a "diagnosis-course" will be shown by the example of a course "axial symmetry".
For the presentation of the dynamic learning-process we use the active character of the concept "schema". This concept describes the interaction of human being with the environment.

1. CONCEPT AND SCHEMA

The human cognitive apparatus is able to store experiences made by connection with appearances. It processes these experiences at the moment it has a new contact with the appearances.

For a given content it can recall parts, elements, attributes and also the relationships between these.

While interacting with parts of the environment, the human understanding learns to select specific features. This selection is governed by his experiences. Thus, the structure of the appearance, which is recognized by the human being, is the concept, the rule for the activities of selection of human understanding. And this rule for systematic activities of selection, fixed by the concept, will be called "schema" (Bussmann 1981, laying out I.Kant).

The human being forms a concept, if he recognizes the essential features of an appearance.

Using the notion of "active schema" it is argued

- that concepts are formed in mental interaction with parts of the environment and that, therefore, the structure of the environment guides the formation of cognitive structures;
- that concepts and schemata are connected in a certain way: the development of concepts and thereby the changes of understanding features characterizing a coherence cause a change of the schemata corresponding to these coherences (Rumelhart and Ortony 1977);
- that schemata have the function to provide the concept with an image (Bussmann 1981). That means for example:

1) schemata help, by the contact of senses with the environment, to construct the perception (Neisser 1976),
2) schemata provide plans of behaviour for solving problems (Schnotz 1979),
3) schemata make possible the mediation of a conception, for example by speech or by drawing activities, and so on (Dörner 1976).

2. ON MATHEMATICAL CONCEPTS

To get a concept of an appearance one contact with the appearance is sufficient. The human being separates the essential, the "general content" (Dawydow 1977). The discovery of what is "formally common" isn't possible in this one contact. Before invariant appearances in different objects can be represented by comparison, respectively by classification, the essential attributes must be separated from the unessential attributes of an appearance.

This interpretation is like the ideas of van Hiele who said (translated from german into english language): What is the sense of speaking about the coherence between the attributes of figures, if the figures themselves don't have a clear shape for the pupil (van Hiele, van Hiele-Geldorf 1979).

Concept that express what is "formally common" of an appearance will also be called "theoretical concepts" (Dawydow 1972).

There is a representation of the theoretical concept in the sensual perception, if one builds a model of it (for example a pyramid).

Mathematical concepts are theoretical concepts. But there is a difference in the teaching of mathematical contents between those concepts, that can be thought as a model from the perception (number) and those concepts, that are constructed in the reality as a model (that means as a generalisation, as an objective representation), for example a drawn model, which could be seen. In both cases a theoretical mathematical concept will be taught. But in the first case the concept must be constructed in the conception. In the second case it exists as a clear model. It requires only sensual perception and not mental construction. But in both cases the pupil uses the understanding of the coherence, in which the concept will be formed. Whilst in the first case the coherence must be constructed by the pupil himself who may already be experiencing difficulties with respect to wrong or incomplete thought-actions (activities of selections on the appearance) in turn leading to a perhaps incomplete concept, in the second case the understanding is easier by direct contemplation.

Therefore in the second case the performance of only one axial-symmetric picture would suffice, to clarify fundamental relations in axial-symmetric pictures. A generalisation and a transfer by the pupil to other axial-symmetric pictures having the same properties is possible very quickly.

3. CONDITIONS THAT MUST BE TAKEN INTO CONSIDERATION FOR THE INVESTIGATION OF THE INDIVIDUAL LEARNING PROCESS

For the formation of concepts the active interaction of the subject with the appearances of the objective reality is important. In our example it's the symmetric model.

If one investigates the learning process, the investigator has his own, subjective representation of it. Therefore we speak of the "representation of the investigator" and his ability to construct himself an almost "objective reality". This statement describes the connection between the possibility of perception and the methods of the investigator to get statements about the object of investigation. The "objective reality" is just the actual part of the environment, which a subject can learn. The learning-process has its foundation in the "contact between the subject and the object", by which the subject constructs or changes his "subjective reality" of the "objective reality".

Diagramm of the given connections:

		parts of investigation of a pupil's learning			
		objective reality	subjective reality	contact between the subject and the object	changes of the subjective reality
the position for the investigator	representation of the investigator	fixation of the environment (course, classroom, etc.)	speech, drawing, action	speech, drawing, action	speech, drawing, action
	activities, methods of the investigator to construct himself an almost objective reality	description of the medium and of the occurrences in the environment: "objective description" of the course	interpretation of the representation by the pupil. Thus the pupil's actions (speech) will be seen in connection with the task.	the pupils 'reports' about the contact with the content, while he describes it (with his speech)	Like the posisition of the investigator interpreting the subjective reality, but here new interpretations must be compared with previous interpretations to assess changes

4. SOME IDEAS ABOUT THE GIVEN RELATIONS IN THE INVESTIGATION OF THE LEARNING PROCESS

In the construction of a course, in which the learning process of a single pupil shall be investigated, the knowledge of the constructor is included. This knowledge is about the

- content and his didactic-methodic formation,
- pupil who will be investigated,
- learning situation and the medium (classroom, teacher, book, etc.),
- changes of the subjective cognition in contact with the objective reality,
- methods, that are possible and practicable to notice a change of the cognitive conditions in the learning sequence.

By planning the learning situation the investigator has to respect, that

- the pupil gets a positive mental attitude for the "whole situation",
- the pupil can concentrate on the teaching-situation,
- the pupil can be active,
- the pupil isn't nervous,
- different pupils need different lengths of time to complete the course.

If these conditions aren't carried out the investigator can't reckon with a controlled situation for the subjective understanding. Then an investigation of the learning process isn't possible.

But if these conditions are carried out, then it is necessary to state the methods helping to promote this sequential process. For example the following methods are possible:

- the pupil will be called upon to react to a given content (contact between the subject and the object). For example he will be called upon to answer a question or to work on a task, so one can see, whether he has understood the task ;
- the pupil will be called upon, to declare aloud, what he thought, and respectively, what he is thinking (so it is possible to ascertain the changes of his subjective reality);
- the pupil will be attended by a teacher ;
- all the actions of the pupil could (for example) be recorded by video.

In spite of all these methods, which could be used in the construction of a course, all the animations and technical instruments should not trouble the pupil in his learning-process. The investigation for thinking aloud must be seen by the pupil as a specific sign of the learning-situation. It should not be seen as a working situation, in which he tries to find out the best answers. To make the work for the pupil easier, it would be better to have a teacher accompanying the pupil. This teacher can help the pupil and he can confirm his actions (also when the actions are false). But the teacher should know his limits in the diagnostic-situation, that means, that his help should be encouraging.

5. THE CONSTRUCTION OF A "DIAGNOSIS-COURSE"

By the construction of a diagnosis-course it is necessary to enable a learning-process in the diagnosis situation. Also for the observation of the pupil it is important to give him enough time for handling.

In this part of the paper the conditions for the construction of each step of the course will be shown. Thus, every step will be shown generally and clearly illustrated by example of a course of "axial-symmetry" in the 5th grade.

I. In General: Activation of the pupil's knowledge of content to make possible the embedding, the internalization, the differentiation, the change of thoughts.
a) The pupil gets direct contact with typical models or symbols of the content.

Diagnosis-course: The pupil has to compare different axial-symmetric pictures and he must try to see the connections. We want to realize his representation of axial-symmetric pictures.

b) In General: The pupil shall use his previous knowledge actively in the handling of new tasks. These tasks will guide the pupil in the understanding of the next problem.

Diagnosis-course: The pupil will confront the problem to draw axial-symmetric figures. An original-triangle and the axis are given. He will draw "free-hand" the reflected image.

II. In General: Production of a contradiction, for example between the result of handling the problem and the knowledge about the coherences of the action.

He could be given an algorithm for the task and find the solution that he could not work out for himself. Subsequently he will be asked to solve the task by his own method. The pupil thus knows the correct solution, but he doesn't know how to get it.

Diagnosis-course: With"folding" or "reflecting" the pupil will be shown a technique for the construction of the exact reflected image. When the pupil has produced the reflected image twice with the learned new technique, and if he recognizes and names the particularities of the axial symmetry, now he will try to get a reflected image with the "geometric-triangle", without the learned technique.

III. In General: The pupil will be asked to clarify his contradiction. That means he should be motivated so well that he wants to find out the coherences that are not clear.

Diagnosis-course: In three exactly drawn "axial-symmetric figures" the pupil now recognizes the regularity of the coherence. For that purpose he gets hints by steps about the position of the picture-figure in comparison with the axis and original-figure.

IV. In General: By the previous step one can see, whether the pupil knows the coherence, whether he finds out or does not find out the coherence, in spite of the given prompts. This will be clear if the pupil has to solve a task of this special coherence.

Diagnosis-course: The pupil will translate the recognized geometrical coherences into a technique of construction with the"geometric-triangle".

V. In General: Like in the school-lesson the pupil will now be told the coherences looked for. This will either confirm the student's solution or require the student to recognize the coherences contained within the solution. These pupils now must recognize the description of the solution.

Diagnosis-course: The rules of axial symmetry will be told to the pupil.

VI. In General: Now it will be shown, whether the pupil has understood the mentioned rules. In that, he must solve corresponding tasks. Subsequently he will be shown the steps to get the solution. Then the pupil makes these steps by himself.

Diagnosis-course: It will be shown to him, how to get the image-point to a given original-point by help of the geometric triangle. Subsequently the pupil exercises this technique on other geometric figures.

VII. In General: It will be observed, how the pupil can transfer this new knowledge to other problems of this content. So one can see for example, whether the pupil changes the learned technique and goes back to an older one.

Diagnosis-course: For example the pupil must construct the axis to a given original and a given picture (reflected image).

VIII. In General: At the end of the course it will be shown, whether the pupil has fixed or changed his realisation in comparison with the beginning of the course.

Diagnosis-course: Given a triangle and an axis, the pupil must construct the "reflected image".

6. THE INVESTIGATION OF PREVIOUS KNOWLEDGE FOR FINDING OUT CHANGES IN THE LEARNING-PROCESS

For recording the subjective reality of the pupil, i.e. the cognitive representation of certain contents, before the course (for example of axial-symmetric fiquares or right angles) it is necessary to gather tasks representing the most important coherence of the course. With these tasks it will be possible to test the pupil's previous knowledge about these coherences. The representation of knowledge must be in the same way like in the later following course (for example: thinking aloud or drawing). A comparison of the two states then is easier. Moreover it's necessary to present these tasks of previous knowledge in different contexts. So one can see, whether even this context of the coherence is important for the pupil's perception of the coherence.

7. ON SOME DIFFICULTIES IN THE INVESTIGATION

1. The speech is not the direct expression of the actions of thinking. Relative to the schema concept given before, one can say that

- words describe general parts of schema,
- words describe an appearance in a way like one who speeks the words understands them,
- one needn't use words for every action of thinking.

An interpretation of the words of the pupil can lead to the integration of subjective ideas of the investigator. A possibility for more objective interpretation of the words would be to see the action in conjunction with the words.

2. If a teacher attends a single pupil, special dependences can arise. For example the aid of the teacher produces an indirect guiding and so it leads to a turning off from the pupil's own productivity to a falsification of his thoughts.

   In an investigation recorded by video, the teacher's activities are reviewed and the following activities of the pupil are interpreted.

3. Very often the situation of the diagnostic-course is still artificial for the pupil. Therefore the results of such investigations can only be transfered to normal classroom conditions with reservations. In the classroom distractions are greater. Changed social conditions would influence the learning process. One can say that these effects can be investigated better after an exploration of the single pupil.

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B

PROPORTION ET PRODUIT
ALGEBRE, FONCTIONS, NOMBRES RATIONNELS
DANS L'ENSEIGNEMENT SECONDAIRE.

PROPORTION AND PRODUCT
ALGEBRA, FUNCTION, RATIONAL NUMBERS
IN SECONDARY EDUCATION.

MESURE DU VOLUME : DIFFICULTES ET ENSEIGNEMENT DANS LES PREMIERES ANNEES DE L'ENSEIGNEMENT SECONDAIRE

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Measure of volume :
Difficulties and teaching in the first years of secondary school

This work was done within the general framework of multiplicative problems (Vergnaud 1978, Rouchier 1980). From this point of view, it is necessary to distinguish two kinds of problems : problems about isomorphism of measure (proportionality), problems about product of measures (areas, volumes, etc...). A first study (Vergnaud 1978) showed that there was a lot of difficulties in calculating with simple volume formulas. After focusing our attention on proportionality problems (Rouchier 1980) we performed three complementary studies about volume. The first one consisted of a series of clinical interviews with 80 children of secondary level from 11-12 years of age to 15-16 years. There were 20 children (10 boys, 10 girls) from each class level. The main result of this work is the fact that a large majority of children don't know the main properties of formulas, i.e. linearity with respect of one dimension, trilinearity with respect of the three dimensions. In fact these aspects of volume are difficult and they are not developped in the school-books. An analysis of school-books showed that they don't present this very important property of formulas (from a conceptual point of view) of reflecting the main characteristic of volume : its homogeneity and dependance with respect of dimensions. It is not our purpose to offer here an alternative and better way of teaching volume, but in a third part of our work, we developped and realised in the classroom a set of didactical situations. Our objective was to determine how it was possible to give some meaning, in the classroom situation, to the most important acpects of volume : measure properties, reflecting geometrical properties for the formulas. We shall give some informations about this part of our work.

Depuis "toujours" ou du moins depuis la généralisation de l'enseignement obligatoire, l'enseignement des notions d'aires et de volumes, autrement dit l'enseignement des formules relatives aux objets usuels : triangles, rectangles, parallélépipède, prisme,... et des systèmes d'unités a fait partie des programmes scolaires. Aires et volumes sont donc partie intégrante du "savoir compter" qui est un objectif de l'école. Pendant longtemps, les manuels anciens en font foi, l'enseignement des volumes s'est limité à apprendre quelques formules simples en même temps que quelques problèmes types d'application de ces formules : place occupée par un corps, capacité, contenance, etc...

Il ne semble pas que ces objectifs aient beaucoup changé. Nous avons conduit une analyse des manuels actuellement utilisés dans l'enseignement secondaire français, analyse qui fera l'objet d'une publication, et constate que, autant par la place occupée, que par le type de problèmes rencontrés, il n'y avait pas eu, à quelques exceptions près, de changement dans le projet d'enseignement relatif au volume.

Notre intérêt pour le volume se situe dans la perspective qui nous occupe depuis plusieurs années, celle de l'étude des problèmes multiplicatifs, essentiellement à travers deux aspects fondamentaux, celui de l'isomorphisme de mesures et celui du produit de mesures (Vergnaud 1978, Rouchier 1980). L'étude physico-mathématique des volumes les constitue en objets pour plusieurs théories mathématiques : en géométrie on classe les invariants de forme et de disposition spatiale, en théorie de la mesure (ou plus largement en théorie des grandeurs) on s'intéresse à des invariants numériques et aux manières de les réaliser. Les deux aspects ne sont pas indépendants, certaines formules de calcul des volumes font largement appel aux caractéristiques géométriques de ces derniers. Ainsi une formule comme V = Sxh pour le prisme condense et abstrait plusieurs aspects fondamentaux relatifs à la forme des prismes et à leur dépendance par rapport à leurs éléments constitutifs : que se passe-t-il par exemple quand on double ou on triple l'aire de la base ou de la hauteur, ou les deux à la fois, etc ... ? Construire et proposer une formule est donc bien le terme d'une mathématisation très riche aussi bien du point de vue du concept de volume que du point de vue d'autres concepts "utilisés" pour la circonstance : variable, linéarité, bilinéarité, trilinéarité, aire, ... Nous avions pu constater lors d'une enquête (Vergnaud 1978) que la differenciation longueurs, aires, volumes ne se mettait en place que tres lentement. Il etait donc nécessaire de mener une recherche complémentaire à plusieurs niveaux pour évaluer les difficultés relatives des notions qui interviennent dans la constitution du concept de volume et cela dans sa construction même. Dans un premier temps nous nous sommes limités à un aspect, que nous appelons arithmétisation, celui de la construction des formules et de la dépendance qu'elles expriment.

Nous avons construit une épreuve à la fois sur la base des résultats de l'enquête préliminaire (Vergnaud 1978) et d'une analyse de l'arithmétisation du volume. Cette épreuve était composée d'une série de problèmes, certains requérant l'utilisation directe des formules, d'autres leur utilisation indirecte (par composition de rapports). Cette épreuve a été soumise à 80 enfants du premier cycle de l'école secondaire (collège) à raison de 20 enfants (10 garçons - 10 filles) par niveau (le collège comporte en France 4 années ou niveaux et accueille des enfants de 11-12 ans à 15-16 ans) en passation individuelle. La consigne était exprimée oralement par l'expérimentateur, l'ordre de succession des problèmes étant choisi parmi plusieurs possibles définis a priori et déterminés au moment de l'expérimentation par la réussite des élèves. L'épreuve a eu lieu en mai-juin de l'année scolaire 1979-80, année de mise en place de nouveaux programmes de mathématiques à l'école secondaire : les enfants de la première année, classe de 6e, suivaient ces nouveaux programmes (selon lesquels le volume est enseigné en deuxième année, classe de 5e), les autres enfants suivaient les anciens programmes selon lesquels le volume était enseigné en première année, classe de 6e. Ainsi un groupe d'enfants n'avait pas suivi d'enseignement systématique du volume. Il faut néanmoins noter que la plupart des maîtres de l'école primaire apprennent les formules du cube, du parallélépipède et parfois du prisme aux enfants.

Les résultats de cette épreuve feront l'objet d'une publication dans laquelle seront détaillés problèmes, réussites et procédures utilisées. Nous ne donnerons ici que deux exemples.
Les enfants avaient à résoudre l'un des deux problèmes suivants :

Problème 5 : Monsieur Dupont a un aquarium assez petit dans sa cuisine et un grand dans son salon. Celui du salon est 2 fois plus long, 3 fois plus large et 2 fois plus profond que celui de la cuisine. Combien de fois celui du salon est-il plus grand que celui de la cuisine ?

Problème 5bis : Combien faut-il que je te donne de cubes pour construire une boîte pleine (comme une boîte de sucre) de 3 de large, de 4 de long et de 2 de haut ?

Le problème 5 était donné aux enfants qui avaient réussi aux deux premiers problèmes (dans lesquels il s'agissait de calculer le volume d'une boîte parallélépipédique et d'estimer le volume de la pièce dans laquelle avait lieu l'expérimentation). Le problème 3 était une question qualitative et le problème 4 consistait à solliciter à nouveau une estimation du volume de la pièce. Le problème 5 bis était donné aux enfants ayant échoué lors de la première partie : 39 enfants sur 80 ont passé le problème 5 (1 en 1e année, 13 en 2e année, 11 en 3e année, 14 en 4e année), 17 ont réussi (1 en 1e année, 4 en 2e année, 4 en 3e année, 8 en 4e année). Un certain nombre d'enfants ne composent pas multiplicativement les rapports et doivent donner des valeurs virtuelles aux dimensions de l'aquarium le plus petit. Quant aux enfants qui ont échoué, les deux grands types de réponse sont, soit la composition additive des rapports, soit le recours au rapport "moyen" (2,5) ou au rapport "modal" (2).
41 enfants sur 80 ont passé le problème 5 bis (19 en 1e année, 7 en 2e année, 9 en 3e année, 6 en 4e année). Il y a assez peu d'enfants des 2e, 3e, 4e années qui échouent, alors qu'en 1e année on trouve 13 échecs contre 6 réussites.

les enfants qui échouent utilisent des procédures de type périmètre (composition additive de longueurs) ou de type surface (composition additive de produits de deux longueurs).

Un autre problème de composition des rapports, le problème 7 : "Voici un L fabriqué avec des légos (il contient 4 cubes) et en voici un autre. J'ai doublé la longueur, la largeur et l'épaisseur (on montre le grand L et on le cache vite). Combien y a-t-il de légos dans le second ?" n'a été réussi que par 9 des 57 élèves qui ont eu à le résoudre.
Ainsi on a pu constater que, d'une part, une fraction non négligeable des enfants de l'école secondaire ne savait pas utiliser les formules permettant de calculer le volume d'un objet de forme parallélépipédique, d'autre part, qu'une proportion très importante d'entre eux ignorait le sens profond des formules de type volume et ne savait pas utiliser la dépendance par rapport aux mesures de longueur qu'elles expriment.

Il y a donc des aspects du concept de volume qui ne sont pas maîtrisés à la fin de l'enseignement secondaire. Cela est certainement dû aux difficultés même de ce concept donc au fait que les enfants ne peuvent pas le maîtriser après une seule rencontre, mais cela est aussi dû au fait que l'enseignement et l'exposition classique du volume sont réduits à l'exhibition de quelques formules. L'analyse des manuels montre que les aspects que nous avons soulignés plus haut sont assez généralement ignorés. Or nous avons toutes les raisons de penser que les manuels reflètent assez correctement la conception que l'enseignement se fait de l'objet mathématique volume donc de la conception qui est présentée aux élèves. Le constat que nous avons dressé plus haut n'est donc pas fortuit, il atteste un certain état de l'enseignement du volume et de ses difficultés.

Dans la méthodologie traditionnelle de la recherche en éducation, on chercherait à conduire des expériences comparatives avec pré-test et post-test, expériences dans lesquelles on ne sait en général pas contrôler les facteurs qui déterminent la conception que les élèves développeront du volume comme objet mathématique. On peut estimer aussi que des tests aussi élaborés soient-ils ne peuvent permettre de saisir tous les aspects propres à une mathématisation : action, formulation, validation. Il est donc nécessaire de construire des expériences didactiques rigoureuses et qui permettront de comprendre comment se construit le sens du volume à travers une série de situations. Nous avons élaboré une suite de leçons qui ont été réalisées dans plusieurs classes du niveau de la 2e année de l'école secondaire (classe de 5e), en 1980 et 1981.

La conception générale des situations faisait une place essentielle à plusieurs aspects du volume. D'abord,il faut montrer que le concept de volume se construit en réponse à un certain nombre de problèmes, par exemple la comparaison de corps de nature physique différente (liquides, solides pleins ou creux), qu'il s'agit d'une grandeur,donc qu'elle peut s'ajouter et se fractionner. Ensuite il faut l'analyser dans ses aspects dimensionnels en construisant les formules qui permettent de calculer les volumes simples. Par exemple, la formule du parallélepipède $V = a \times b \times c$ résume le remplissage d'une boîte par des cubes unités et représente un invariant de nature géométrique. On désigne ainsi à la fois l'invariance de la formule par affinité sur chaque variable et par homothétie sur les trois variables : propriété de linéarité et de trilinéarité. De même la formule du prisme $V = S \times h$ représente la dépendance bilinéaire par rapport à deux grandeurs différentes, aire et longueur. Enfin, il faut rencontrer des problèmes pour lesquels des calculs de périmètres, d'aires et de volumes permettent de discriminer ces trois grandeurs et analyser leurs relations mutuelles.

Nous pouvons donner quelques indications sur la construction d'une formule pour exprimer le volume d'un prisme à base rectangulaire. On va demander aux enfants de prévoir l'ordre des volumes de quatre prismes de même hauteur (8 cm) construits sur des triangles A, B, C, D dont les hauteurs sont respectivement 5, 5, 5, 10 et les côtés correspondants 6, 6, 12, 6 ; puis de construire un prisme de base rectangulaire de volume double à celui construit sur B et un prisme de base triangulaire (avec un triangle de hauteur 10) de volume quadruple du prisme construit sur B. Après une vérification qualitative, on peut comparer les modèles utilisés et émettre des hypothèses sur la dépendance du volume par rapport à l'aire du triangle de base et par rapport à la hauteur. Ces hypothèses permettent de constituer un premier tableau:

S \ h			8	
15			120 b	

avec lequel on trouvera les volumes en cm^3, pour tous les prismes construits,à partir de celui du prisme B (moitié d'un prisme à base rectangulaire, donc d'un parallélépipède, volume qu'on sait déjà calculer). La manipulation de ce tableau équivaut sur le plan de l'action à celle de la propriété : si une quantité est proportionnelle à deux quantités indépendantes, elle est proportionnelle à leur produit. Cette propriété fondamentale pour établir des formules a déjà été rencontrée à propos de l'aire du triangle. On va donc proposer de calculer les aires des différents triangles, d'en envisager de nouveaux et de construire un tableau à double entrée analogue au précédent. Ce dernier sera complété à son tour et on constatera que des triangles "différents" (hauteur et côté differents) se retrouvent, du fait qu'ils ont la même aire, dans la même case du tableau du volume. Pour le calcul de ce dernier, c'est bien l'aire qui compte, ce que résume la formule V = S x h.

Nous donnerons quelques éléments d'analyse des situations didactiques que nous avons réalisées. Elles ne constituent pas à proprement parler une proposition alternative mais d'abord un moyen d'étude des conditions didactiques de la construction du sens du volume. Il nous apparait neanmoins qu'il est possible de proposer le volume comme une mathématisation, comme le produit d'une théorisation, alors que dans les conceptions traditionnelles il paraît relégué comme une application élémentaire. Une certaine idéologie y trouve son compte, une idéologie de la fausse simplicité des formules et de leur utilisation.
La construction correcte du volume et de l'aire passe au moins par un enrichissement et une diversification des situations didactiques que les enfants ont à connaître. Cet enrichissement et cette diversification demandent des études complémentaires à celle que nous avons menée. En particulier, les travaux de J. Rogalski sur la conception de l'aire montrent des difficultés analogues à celles que nous avons relevées. Il est nécessaire d'étudier les conditions d'une meilleure discrimination surface-volume ainsi que celles d'une reprise et d'un approfondissement du concept de volume dans les années ultérieures de l'école secondaire, reprise qui n'est pas effectuée dans l'enseignement actuel, par construction d'autres formules (pyramide, sphère), approximation, développement de l'aspect fonctionnel des formules. Nous pensons que c'est ainsi qu'on pourra contribuer à une meilleure connaissance des caractéristiques des situations didactiques propres à favoriser l'acquisition du concept de volume.

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ACQUISITION DE LA NOTION DE DIMENSION DES MESURES SPATIALES DE LONGUEUR ET SURFACE

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The properties and notions concerning the spatial measures: length, area, volume are multiple, complex, with strong internal relationships. So the spatial measurement constitutes in itself a "conceptual field". The acquisition by children and adolescents of the fundamental notions in this field is a complex and long-time process. Studies with students from 4th to 7th grade (10 to 15 years old) show that the appropriation of the central concept of "dimension which differenciates and organizes the three spatial measures is not achieved at the end of the so-called "cycle d'observation".
Students, having to use additive properties of area to solve a task involving its multiplicative dimensional invariant, attested many difficulties - even for familiar figures such as squares, parallelograms and triangles. They frequently use a "linear model", appropriate to length measure, specially in the case where the area-unit is linked to length-unit, as cm^2 is.
The results ask questions about the actual limitation in time and content of the teaching of these notions, and the inadequacy with the conceptual difficulty of the notions involved in this field.

Le champ conceptuel des mesures spatiales est complexe à la fois dans son organisation et dans son appropriation cognitive par l'enfant et l'élève. Les recherches ont ainsi montré que les premières conservations sur la longueur et la surface sont assurées vers 7-8 ans (Piaget et al., 1948a, 1948b) mais que la différenciation périmetre-surface n'était pas acquise avant 12 ans (Vinh Bang, 1965), et que les propriétés du volume étaient d'une complexité encore plus grande (Piaget et al., 1948b; Vergnaud et al., 1979).

En particulier les notions relevant de la "dimensionalité" relative des mesures spatiales (longueur: dimension 1, surface: dimension 2, volume: dimension 3) font l'objet d'un processus d'acquisition de très longue durée, dans lequel l'enseignement joue un rôle important mais encore mal contrôlé. Les bilans effectués à la fin de l'enseignement élémentaire (I.N.R.P., 1977; N.A.E.P., 1980), l'étude des conceptions erronées des élèves sur la surface (Hirstein et al., 1978), l'analyse de procédure de calcul de volume (Vergnaud et al., 1978, 1979) témoignent de la confusion entre les propriétés dimensionnelles des longueurs (périmètres) et des surfaces des figures d'une part, entre celles des surfaces et des volumes des objets d'autre part.

L'analyse des relations entre les quantités spatiales - éléments de connaissance du monde, et les opérations de calcul - résultat d'une mathématisation opératoire, montre la complexité des opérations cognitives à mettre en oeuvre et la complexité des relations entre les différentes notions "physico-spatiales" à coordonner (Rogalski, 1979). On doit donc s'attendre à ce que la différenciation des propriétés des différentes mesures spatiales soit longue et difficile, à ce qu'il y ait des interactions importantes avec les caractéristiques propres des figures ou objets considérés, à ce que le "mode" avec lequel on opère sur les mesures intervienne également: une surface comme "étendue à peindre", (ou un volume comme "capacité) est exprimée comme quantité simple avec une unité unidimensionnelle (le nombre de pots de peinture par exemple); une surface exprimée avec des unités rapportées aux longueurs (le cm^2 par exemple) est implicitement considérée dans son caractère bidimensionnel.

Une étude sur la "dimensionalité" des longueurs et surfaces a été conduite avec des élèves de CM_1 à 4ème pour vérifier ces hypothèses et analyser plus précisément les opérations cognitives en jeu. Le paradigme expérimental est proche de celui des conservations: des figures familières (carrés, parallélogrammes, triangles, cercle) sont transformées par des similitudes de rapport simple. Les questions sont posées sur la mesure de la figure transformée pour mettre en oeuvre la propriété suivante: le rapport mesure transformée/mesure initiale est indépendant de la figure, c'est un invariant dimensionnel qui vaut r pour la longueur, r^2 pour la surface r^3 pour le volume.

"UNIDIMENSIONALITE" DE LA MESURE LINEAIRE

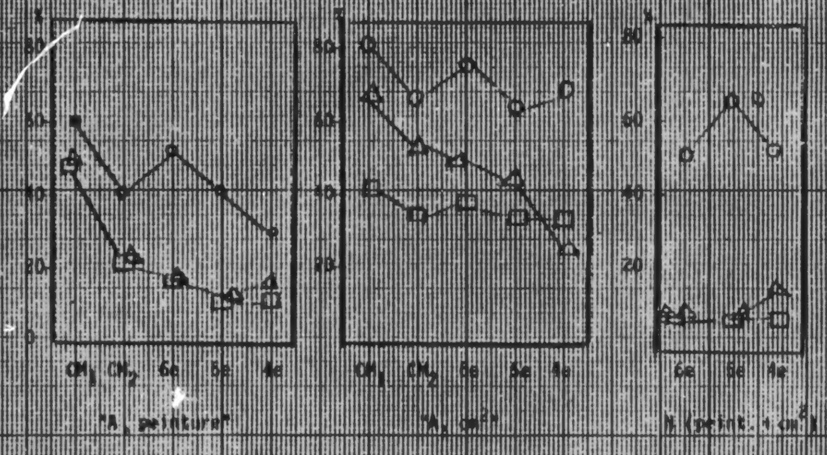
La disponibilité d'un calcul additif, possible pour des figures à bord rectiligne (carrés, ...) est présente pour la plus grande majorité des élèves. Globalement un élève sur deux, au CM_1, trois sur quatre en fin de 5ème font des réponses respectant l'unidimensionalité des longueurs, et le décalage entre "linéarité rectiligne" et linéarité curviligne" est faible (de 20% en CM_1 à 10% en 5ème). Cependant la fiabilité opératoire de cette notion connait une évolution importante; très faible en CM_1, on peut la considérée comme assurée pour les trois-quarts des élèves en fin de cycle d'observation, pour les données numériques simples choisies pour ces épreuves. On peut dire que la structuration de cette notion fondamentale d'unidimensionalité de la longueur s'étand sur les quatre années du cycle moyen et du cycle d'observation.
Une comparaison des résultats d'ensemble confirme que les questions de dimensionalité de la surface sont plus complexes que celles sur la longueur, et celles-ci moins difficiles que pour le volume, les décalages des réussites moyennes étant notables.

BIDIMENSIONALITE DE LA MESURE-SURFACE

La bidimensionalité de l'aire est une notion dont l'acquisition est très difficile et reste largement inachevée pour beaucoup d'adolescents. Le domaine de validité des opérations que l'élève peut engager avec succès pour résoudre des problèmes liés à cette dimensionalité est limité, l'interaction avec le mode d'expression de la mesure est forte et la structuration elle-même des situations-problèmes intervient de manière complexe. Ainsi des figures comme les rectangles et parallélogrammes définies par le croisement de deux directions indépendantes, et pavables par la même opération, "supportent" des opérations de caractère bidimensionnel, alors que des figures comme les triangles, dont la description comme le pavage utilise trois directions non indépendantes, sont traitées spécifiquement, avec une utilisation notable du nombre de côtés comme critère pertinent pour les réponses sur la surface.

Enfin une figure comme le cercle, pour laquelle il n'existe pas de passage direct du pavage par des figures semblables au calcul de la surface transformée, est l'objet d'un transfert massif du "modèle linéaire".

Les graphiques suivants montrent l'évolution complexe de l'utilisation du "modèle linéaire"(Fig.1) et de l'utilisation de l'"invariant dimensionnel" (Fig.2) selon les figures, le mode opératoire et la structuration des situations-problèmes. ("N" est davantage structurée que "A" et les résultats n'y changent pas avec le mode opératoire). Les signes □ et △ représentent les résultats respectivement des carrés et parallélogrammes et des triangles (équilatéral et obtus).



"A, peinture"   "A, cm²"   N (peint. + cm²)

Fig.1 "MODELE LINEAIRE"

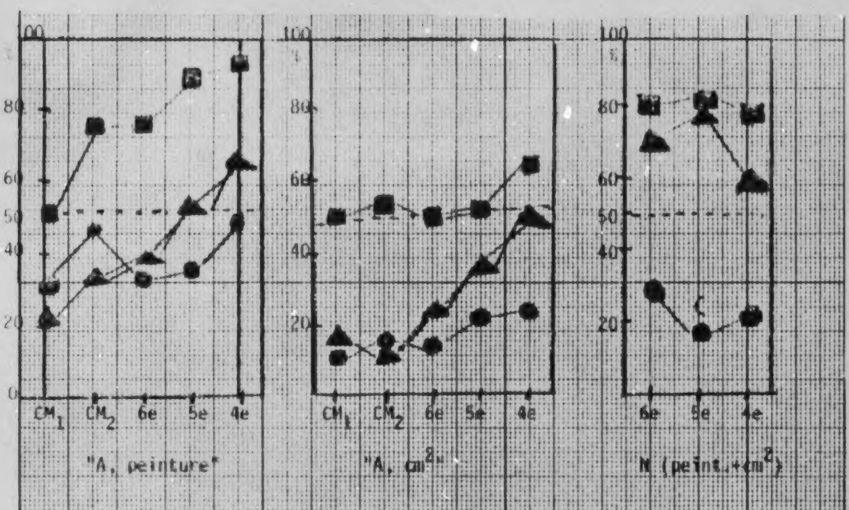


Fig.2 "COEFFICIENT BIDIMENSIONNEL"

CONCLUSION

Il est possible de mobiliser les opérations cognitives fondamentales pour le passage de l'additif au multiplicatif, pour les différents modes opératoires et les figures dans un domaine assez large pour permettre ultérieurement - par combinaison et continuité - de construire l'invariant dimensionnel de la surface, différencie de celui de la longueur. Mais les représentations spatiales nécessaires, et les notions constitutives de la bidimensionalité de la surface ne sont pas appropriées par les adolescents: les opérations sont peu fiables et peu généralisables. Ainsi, une rationalité certaine est présente tôt dans les réponses des élèves, des acquisitions notionnelles importantes ont lieu, en particulier lors du premier enseignement systématique (CM_2), mais le travail sur la différenciation et la coordination des mesures de longueur et de surface n'est pas suffisant, ni assez prolongé eu égard à la complexité conceptuelle des mesures spatiales.

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PROPORTION ET EQUILIBRE DE LA BALANCE: UNE EXPERIENCE D'APPRENTISSAGE DE LA PROPORTION INVERSE

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We have tried to enable children of the 6 grade (11-12 years old) to acquire the notion of proportion through a specific learning method. The children of that grade and of that age know how to operate proportion calculation within a situation similar to those encountered in the educational environment, but it is not the case when the problem is given out of the scholastic activities frame. The problem we have thus kept as an extra-scholastic one is the equilibrium of the balance scale ; the rule follows the relation of the inverse proportion which links the weights W and W', and the distances D and D' in between. We have the equilibrium if $W / W' = D' / D$.

The training method is based on the hypothesis that there does exist an identity of related structures between the qualitative answer of inverse correspondance between weight and distance (the farther away it is, the less weight it has) and the answer which quantifies by means of operators (it is 3 times less weight, then I need to put it 3 times farther away). We employ a technique where the child must discover the rule of the game (rule of the inversion of the operator). And it is only at the last step that the four terms of the proportion are introduced.

This method prooves to be truly efficient when transfered to the scale problem, one month later after the training sequences. We get answers of proportion given to the scale problem by all the children of the experimental group, versus one child of the check sample.

L'enseignement de la proportion est introduit dans les programmes scolaires français au niveau du CM2 (enfants de 10-11 ans). Au-delà de la classe de 3ème (14-15 ans), cette notion est considérée comme acquise.

Or, de nombreux travaux s'inspirant des observations de Piaget et ses collaborateurs montrent que l'acquisition de la notion de proportion ne va pas sans soulever quelques difficultés. Les expériences auxquelles nous faisons allusion ont utilisé les épreuves qu'avaient employées Inhelder et Piaget (1955). On constate que seulement la moitié des sujets de 15 ans (Jackson, 1965), ou de 17 ans (Martorano, 1974) donnent des réponses de proportion dans ce type d'épreuve. Les sujets de 13 ans (Lee, 1971) ainsi que la majorité des sujets adultes (Lovell, 1961) fonctionnent, toujours dans ce type d'épreuve, au niveau opératoire concret (n'établissent pas de quantification).

Il faut remarquer cependant que lorsque le problème de proportion intervient dans une épreuve familière aux enfants, à savoir présentant une certaine similitude avec les exercices scolaires, les résultats obtenus sont très différents de ceux rapportés par les auteurs cités plus haut. Sur 74 élèves de 11-12 ans des classes de 6ème, nous avons en effet trouvé 67 enfants qui réussissent au moins un item au moyen de calculs proportionnels dans un problème proche de ceux rencontrés en classe, alors que parmi ces mêm enfants, 36 sont incapables, une semaine après, d'établir une quantification proportio nelle à un problème physique utilisé par l'équipe de Piaget (Inhelder et Piaget, 1955, Vinh Bang, 1968, Piaget, 1974), et assez différent des problèmes scolaires.

Notre propos ici n'est ni de nous interroger sur la compatibilité de tels résultats avec la théorie de Piaget, ni non plus de soulever la question de la portée de l'enseignement de la proportion ; mais nous voulons essayer de comprendre comment se construit chez l'enfant la notion de proportion. Pour cela nous avons effectué certaines expériences (Reinisch, 1980), mais les résultats obtenus nous ont conduits à envisager l'emploi d'une autre méthode que la méthode transversale, utilisée par nous jusqu'alo et qui paraîtrait plus adaptée à résoudre notre problème. Nous avons mis au point une procédure d'apprentissage, susceptible de faire acquérir la notion de proportion à l'enfant, et ainsi de nous renseigner sur les étapes possibles de l'évolution de cett notion. Pour élaborer cette procédure, nous avons défini d'une part des étapes théoriques, à partir d'une analyse de la tâche, et d'autre part inscrit ces étapes dans une perspective évolutive : la succession de ces étapes, construite à partir du nivea réel des enfants, devait les conduire à la proportion.

CONSTRUCTION DE LA PROCEDURE D'APPRENTISSAGE

L'épreuve que nous avons choisie pour examiner l'acquisition de la notion de proportio est un problème physique où intervient une loi de proportion inverse. Cette épreuve est celle de l'équilibre de la balance; la balance est composée d un fléau en équilib en son point médian. Si P et P' sont les poids d'un côté et de l'autre du fléau, et D et D' les distances auxquelles ils sont respectivement placés, on a l'équilibre si $P / P' = D' / D$.

Dans une première expérience effectuée (Reinisch, 1980), nous proposions à l'enfant de créer un déséquilibre , puis de rétablir l'équilibre, à plusieurs reprises, et de différentes manières. Nous avons pu analyser les résultats selon un modèle proche de celui utilisé par Siegler (1976); on peut décrire les étapes hiérarchisées suivantes: découverte d'une dimension (le poids) puis d'une autre (la distance) et enfin coordination des deux (correspondance inverse entre poids et distance). Les enfants se conformant à ce dernier type de règle verbalisent ainsi: "J'ai moins de poids, alors je dois le mettre plus loin". Notre hypothèse est que la quantification proportionnelle se construit à partir de cette notion de correspondance inverse, mais cette construction présente pour l'enfant quelques difficultés: "La distance est plus petite, je le vois, mais de combien plus petite, c'est bien autre chose". Cette remarque d'une enfant examiné par Vinh Bang (1968) rend compte de la frontière qui sépare, pour la proportion, la notion qualitative, de sa quantification.

Le problème paraît donc être le passage de la notion de correspondance inverse (qualitative) à celle de proportion inverse (quantitative). La procédure d'apprentissage que nous avons mise au point tente de réaliser ce passage.

Nous avons, dans un premier temps, effectué une analyse de la proportion, du moins telle qu'elle apparaît dans les réponses que donnent les enfants au problème de l'équilibr de la balance. Les enfants donnent en général des réponses de type: "Il y a 3 fois moin de poids, alors je dois le mettre 3 fois plus loin". A partir des valeurs du poids, on doit abstraire l'opérateur, que l'on doit transférer (avec une inversion) sur la distance. On a donc un premier niveau, celui des données numériques. A un deuxième niveau, on a déjà un traitement sur les états, c'est un premier niveau relationnel. Le deuxième niveau relationnel est celui du traitement de l'opérateur, l'opération qui consiste à inverser l'opérateur obtenu sur les poids. On a donc 3 niveaux, le premier, celui des états, le deuxième, celui des transformations (abstraction des opérateurs), et le 3ème, celui des transformations de transformations (inversion de l'opérateur).

Or, il semble que cette structure relationnelle, bien que complexe, soit très proche d la structure de correspondance inverse qualitative dont témoigne l'enfant quand il donn des réponses du type: "Plus c'est loin, moins il faut mettre de poids". On a mis l'accent sur le passage de cette structure à la structure de proportion proprement dite. l'hypothèse est qu'il s'agit d'une structure relationnelle voisine, le manque résidant dans la quantification (les enfants ayant d'une part la structure de correspondance inverse, et d'autre part les données numériques dont ils ne savent que faire). L'effort de l'apprentissage consiste à introduire la quantification, à travers la même structure compatible à la fois avec ce que savent les enfants dans l'épreuve de l'équilibre de la balance (correspondance inverse), et la structure relationnelle finale, complète, de la proportion inverse.

La technique d'apprentissage est basée sur le principe que l'on va jouer à un jeu, don l'enfant devra trouver la règle. Dans la première séance, le jeu est d'inverser l'opérateur, celui-ci étant fourni à l'enfant. Dans la 2ème, l'enfant doit inverser l'opérateu mais après l'avoir découvert. A la 3ème séance, enfin, le jeu est toujours d'inverser l'opérateur, mais après l'avoir abstrait au moyen d'un calcul à partir des données numé riques disponibles.

1ère séance: On utilise des bandelettes de bristol; on propose à l'enfant, à partir d'u bande de 16 cm./1 cm., de la couper en 2, et on lui demande, au moyen de ces deux morceaux obtenus, de former un autre rectangle qui n'ait pas la même forme que le 1er. L'enfant obtient un rectangle de 8 cm. / 2 cm. On lui demande alors comment est devenu la longueur, et comment est devenue la hauteur. On continue ensuite le même jeu, en recoupant chaque fois la figure obtenue en 2 parties et en effectuant la comparaison 2 à On récapitule enfin (après obtention d'un rectangle de 1 cm. / 16 cm.) en demandant co ment la longueur et la hauteur ont changé, depuis le 1er jusqu'au dernier rectangle ob nu. On recommence ensuite une séquence identique en coupant en 3 parties égales un rectangle de 27 cm. / 1 cm. (opérateur /3) et en 4 parties égales un rectangle de 32 cm. 0.5 cm. (opérateur /4). On termine la séance en demandant à l'enfant quelle était la règle du jeu.

2ème séance: Après avoir averti l'enfant qu'il s'agissait du même jeu, mais un peu différent, on lui propose une feuille de papier non quadrillé sur laquelle sont tracés un rectangle et la longueur d'un autre rectangle, dont la hauteur est à trouver. On lui d mande comment est devenue la longueur, et on l'invite à la "mesurer" par rapport à l'a tre longueur au moyen d'un étalon (on donne à l'enfant de fines tiges métalliques, cho sies aux dimensions de s longueurs et hauteurs des différents items). On demande ensui comment doit augmenter (ou diminuer) la hauteur, et l'enfant doit la tracer , après av effectué les "mesures" au moyen des étalons. La séance comprend 7 items, où les opérateurs choisis sont entiers, et compris de 2 à 5. On demande en fin de séance quelle était la règle du jeu.

3ème séance: Après avoir averti l'enfant que l'on jouerait au même jeu, on lui fournit un matériel prèche de celui de la 2ème séance, mais le papier est cette fois quadrill La mesure se fait donc non plus au moyen d'étalons,mais elle utilise les petits carre de la feuille. Il s'agit d'une mesure absolue, et non relative (comme elle l'était dan la 2ème séance). La séance se déroule suivant les mêmes modalités que la 2ème: on po à l'enfant des questions sur le changement de la longueur, et le changement que devra subir la hauteur, "toujours pour jouer à la même règle du jeu".

EXPERIENCE ET RESULTATS

La procédure d'apprentissage est construite à partir de la structure de correspondanc verse, et introduit la quantification proportionnelle progressivement. Il paraissait essentiel que les enfants témoignent, dans l'épreuve de l'équilibre de la balance, de réponses de correspondance inverse. D'autre part, la proportion est enseignée à l'école, et les enfants n'ont souvent pas de difficultés à donner des réponses de proportion dans des problèmes géométriques, sans pour cela être capables d'établir des relations proportionnelles quand on leur propose un problème extra-scolaire.

Choix des sujets: Nous avons choisi les enfants en fonction d'un niveau maximal de correspondance inverse à l'épreuve de la balance, et d'un niveau minimal de proportion à une épreuve d'agrandissement de rectangles, proche des problèmes scolaires (épreuve proposée par Longeot, 1972). Ces enfants étaient issus des classes de 6ème d'un même établissement scolaire, et avaient entre 11 ans et 11ans 9 mois. 36 sujets ont été retenus répartir en 18 pour le groupe expérimental, et 18 pour le groupe témoin. Ces groupes sont appariés sur la base des résultats au pré-test (balance, et agrandissement des rectangles). L'expérience comprend 3 phases pour le groupe expérimental: pré-test, apprentissage 2 semaines après le pré-test, post-test (épreuve de la balance) 3 semaines après l'apprentissage. Le groupe témoin ne participe pas aux séances d'apprentissage.

Classement des réponses: On peut regrouper les réponses obtenues en 7 catégories:

(1) Enoncé de la loi sous forme qualitative (correspondance inverse)
(2) Essai de quantification additive
(3) Découverte d'opérateur sur une seule dimension (poids ou distance)
(4) Réponse de produit des deux dimensions incomplète ($P_1 \times D_1 = x$)
(5) Réponse de produit complète ($P_1 \times D_1 = P_2 \times x$)
(6) Abstraction et inversion d'opérateur sur exemple précis
(7) Abstraction et inversion d'opérateur avec généralisation

Résultats: On posait à l'enfant une question d'anticipation, on lui demandait ensuite de réaliser l'équilibre, en trouvant à quelle distance il fallait placer un poids donné le poids et la distance étant déterminés de l'autre côté du fléau. On lui demandait alo de justifier le résultat obtenu, et à la fin de l'épreuve, on posait la question: "Quelle est la règle, la loi du système ?". Les résultats obtenus sont les suivants:

TABLEAU I : RESULTATS OBTENUS EN ANTICIPATION

	(1)	(2)	(3)	(4)	(5)	(6)	(7)	N
GE	0	0	0	0	0	16	2	18
GT	5	2	9	0	1	1	0	18

TABLEAU II : RESULTATS OBTENUS EN JUSTIFICATION

	(1)	(2)	(3)	(4)	(5)	(6)	(7)	N
GE	0	0	0	0	0	15	3	18
GT	3	0	5	5	2	3	0	18

TABLEAU III : RESULTATS OBTENUS EN REGLE FINALE

	(1)	(2)	(3)	(4)	(5)	(6)	(7)	N
GE	0	0	0	1	0	13	4	18
GT	1	7	3	4	2	1	0	18

Les résultats indiquent clairement l'efficacité de cette procédure d'apprentissage pou
son transfert à l'épreuve de l'équilibre de la balance. Mais la question reste posée d
la portée et des limites de cette procédure. Il manque à ce travail l'étude de la poss
bilité d'effets à long terme. D'autre part nous ne savons pas dans quelle mesure cette
procédure, construite à partir d'une analyse de la tâche de la balance (inversion d'op
rateurs) est transférable à une tâche voisine, mais différente (comparaison de deux ra
ports par exemple). Nos recherches s'orientent dans cette direction.

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PROCESSES NEEDED BY AVERAGE ABILITY MIDDLE SCHOOL CHILDREN IN THE SOLUTION OF "REAL WORLD" MATHEMATICS PROBLEMS

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Cet article résume sommairement les objectifs principaux et les hypothèses qui sont à la base d'un projet de recherche financé par NSF et concernant les procédés de solution requis par des élèves d'habilité moyenne de l'enseignement secondaire pour résoudre des problèmes de mathématique du monde réel. On y décrit des exemples sur 1) des problèmes utilisés dans le projet APS, 2) des charactéristiques importantes qui influence la façon de résoudre un problème, et 3) des procédés qui sont nécessaires à la solution de problèmes mathématiques de chaque jour. D'intérêt particulier sont les procédés nécessaires durant les étapes de la solution où une réponse n'est pas à donner, comme par example, le developement du modèle (y compris l'introduction de systèmes représentatifs appropriés), le relèvement de l'information et l'évaluation de la solution (ou du modèle). D'autres procédès d'intérêt comprennent ceux qui sont considérés importants dans d'autres domaines de recherche (par example, des problèmes de mots et des problèmes de prise de décision dans une situation médicale, etc.) mais qui ont besoin d'être reinterprétés pour s'appliquer à des solutions de problèmes pratiques.

This paper briefly summarizes some of the major goals and assumptions underlying an NSF-funded research project concerning applied mathematical problem solving for middle school youngsters. The Applied Problem Solving project focuses on ideas and processes that are accessible to average ability middle school children when they try to solve real problems involving substantial mathematical ideas in realistic situations. A rationale for these four foci is given in Lesh (1981).

Goals of the project include:

A. Producing a set of mathematically rich and psychologically interesting problems which involve easy-to-identify, substantive mathematical content, real (or at least realistic) data or problem situations, and realistic problem solving resources--including technological tools (e.g., calculators) or other people (e.g., peers and consultants). Some of the problems require approximately one hour to complete and can be used in small group problem solving sessions as well as with individual students. For these latter types of complex problem solving episodes, videotape protocol analyses will be accompanied by follow-up interviews and by instruments for measuring relevant background information (e.g., prerequisite concept acquisition).

B. Conducting "task analyses" and "idea analyses" for the problems developed in the project, identifying: (1) important problem characteristics which influence problem solving behavior, and (2) processes which are needed in the solution of everyday mathematical problems. Processes of particular interest are those needed at non-answer giving stages of problem solving, e.g., model development (including the introduction of appropriate representational systems), information retrieval, and solution (or model) evaluation. Other processes which are of interest include those which research in other problem solving contexts (e.g., word problems, medical decision making, etc.) have claimed are important, but which appear to require reinterpretation in order to fit applied problem solving situations.

C. Producing evaluation instruments to measure selected: (1) processes (e.g., modeling processes), (2) abilities (e.g., modified versions of several "abilities" identified by Krutetskii (1976) or "disabilities" identified by Lesh (1980); (3) skills (e.g., manipulating equations which involve both number and unit labels--30 miles/hour x 30 minutes = ☐), and (4) understandings (e.g., metacognitive understandings). The project will investigate the predictive value of attitude or ability measures (e.g., self-directedness, creativity, field dependence) which appear to be influenced by content understanding in a particular domain or by metacognitive understandings associated with that area.

D. Refining Saari's (1977) mathematical model of problem solving to account for data resulting from the project's task analyses and underlying idea analyses, or from its analyses of problem solver characteristics.

E. Refining Bell and Usiskin's (Note 1) taxonomy of mathematical uses. Bell and Usiskin's NSF-funded project has produced a classification scheme derived from a logical analysis of ways elementary mathematical ideas are used in everyday situations. The scheme gives a way of organizing mathematical content, together with example problems in each content category--accompanied by references to other sources of applied problems related to the category. The present project will refine Bell and Usiskin's taxonomy by superimposing on their logical analysis a psychological characterization of categories--together with important processes which students must use when they work on problems in each category. These refinements involve redefining some categories, creating new categories, and collapsing or deleting other categories.

PAST RESEARCH WHICH INFLUENCED THE APS PROJECT

Many of the theoretical perspectives for the applied problem solving project evolved during earlier research investigating: (a) the development of spatial/geometric concepts in children and adults (Lesh & Mierkiewicz, 1978); (b) mathematical abilities that are deficient in "learning disabilities" subjects (Lesh, 1980); and (c) the role that various representational systems play in the acquisition and use of rational number concepts (Behr, Lesh, & Post, Note 2).

Because modern psychology attributes a multiplicity of different meanings to the term "information processing" and because many of these interpretations do not fit the theoretical perspectives underlying the APS project, we do not characterize ours as an information processing approach. However, like most information processing perspectives, we do treat the learner as an adaptive system whose interpretation of problems is influenced by internal models as well as by external stimuli. On the other hand, we do not treat mathematics as information to be processed, nor do we treat mathematicians as processors. For us, the mathematician or mathematics student is considered to be a "situation interpreter and transformer," and mathematics furnishes the "conceptual models" (see definition below) for making interpretations and transformations.

Our explanations of cognitive growth tend to be more "organismic" than "mechanistic," with our theoretical constructs bearing closer resemblances to many of Piaget's ideas than to artificial intelligence models (Lesh, 1980; Saari, Note 3). We tend to focus on "tracing the development of ideas" rather than "tracing the development of children," and on "idea analyses" rather than "task analyses" (Lesh & Landau, 1981). We prefer to use modified versions of "related" mathematical systems to model children's primitive conceptualizations of mathematical ideas, rather than using quasi-linguistic models (Lesh, 1980; Saari, Note 3). And, unlike many information processing theories, the APS project is not based on the assumption that chains of productions (i.e., condition-action pairs) are at the heart of most thinking. For example, one might contrast our point of view with that adopted by Newell and Simon in their book, Human Problem Solving (1972):

> The theory to be presented in this book has much more to say about methods and executive organizations than about creating new representations of shifting from one representation to another....

> However, some problems do exist in which the whole difficulty of solution resides in finding the right representation. Once that representation has been discovered, solving the problem becomes a trivial matter. (p. 90)

The APS project is concerned precisely with problems in which the development of an appropriate model is important, and a number of examples will be given during my presentation in Grenoble. We believe these types of problems are typical of realistic situations in which mathematics is used to solve problems. For us, mathematics is the study *of structure*, the content of mathematics consists of *structures*, and to *do* mathematics is to *create* and *manipulate* structures. These structures, whether they are embedded in pictures, manipulative materials, spoken language, or written symbols, are the models that mathematicians and mathematics students use to solve problems.

Definition: A *problem* is a meaningful situation which a student is willing to address, but for which a stable conceptual model is not available.

Definition: A *conceptual model* is an integrated system which includes: (a) an idea which presupposes systematic relationships with other ideas and a system of relations and operations that comprise the formal definition of the concept), (b) a representational system, and (c) a system of processes which contribute to the meaningfulness and usability of the idea.

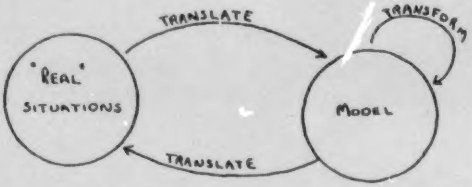


Figure 1

Processes Needed to Use Mathematical Ideas in Real Situations

Figure 1 represents a useful but naive conceptualization of the problem solving process. It is naive because a given model may be associated with several distinct representational systems (i.e., pictures, spoken language, written symbols, etc.), each of which may be "good" for representing some aspects of the problem but "not so good" for representing others. Different aspects of the problem may be represented using different systems, and the solution processes may involve "mapping" back and forth among several systems--perhaps using pictures as an intermediary between the real situation and written symbols, or perhaps using language as an intermediary between pictures and written symbols. In fact, many real world problems occur in a multi-modal form and one of the students' initial problems is to express the data using a single representational system. Examples of these phenomena will be given in my presentation at Grenoble.

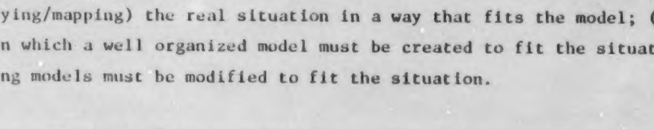
Not only do problem solvers translate back and forth among various representational systems during the solution of a particular problem, they also map back and forth between internal and external versions of these systems. For example, in early stages of the solution of a given problem, a child may draw a photograph-like depictive picture of the problem situation. This depictive drawing may simplify the real situation by leaving out some information, clarifying other relationships by organizing and weighting the information, and reducing memory load. This may allow the child to clarify or reorganize his internal models, and this may in turn allow the child to produce a more refined and realistic "descriptive" diagram.

Unlike most definitions of the term "problem," we do not characterize problem solving as an inability to get from A to B. For example, from our perspective, a mountain climber's "problem" is not so much "to get from the bottom of a cliff to the top" as it is to "understand the terrain." Once the terrain is understood, the activity of getting to the top of the cliff is an exercise, not a problem.

In most puzzles or problems used to study "problem solving," the starting situation and the desired end point are both given. More realistic problems often occur as "ouches" rather than as well defined questions with clearly specified goals (in which the "problem" is to find a set of legal moves to get from the "givens" to the "answer"). Problem formulation is an important stage in real problem solving. In many cases, there is an overwhelming amount of information, all of which is relevant to the problem, and the main difficulty may be to select and organize the information that is "most useful" in order to find an answer that is "good enough." In other cases, not enough information is available, but a usable answer must be found anyway. Or, additional information may need to be identified or generated as part of a solution attempt--the information may not all be given at the start. All of these characteristics of real problems are related to the use of conceptual models as "filters" to select, organize, and interpret information from real situations. This filter always distorts or deemphasizes some aspects of real situations in order to clarify or emphasize others. Consequently, investigating the usefulness of trial "models" is related to a variety of important applied problem solving processes. These include problem formulation processes, modeling processes, representational processes, and solution evaluation processes to investigate the "goodness" or "usefulness" of various trial solutions to a given problem. Again, examples of these and other problem characteristics will be given in my presentation at Grenoble.

Problem solving often requires generating a sequence of progressively more refined problems. Our problems come in two types, or combinations of these two types (Lesh, 1981): (a) problems in which a well organized model is available but the amount of information needed to deal with the situation analytically exceeds the processing capabilities of the individual. In these cases, the problem solving process consists of interpreting (i.e., simplifying/clarifying/mapping) the real situation in a way that fits the model; (b) problems in which a well organized model must be created to fit the situation, or existing models must be modified to fit the situation.

How do our conceptual models differ from more traditional information processing models? Primarily, the differences result from our emphasis on the structure of mathematical ideas, and on the interdependent roles of idea structures, processes which contribute to the meaning and usability of the idea, and the representational systems in which these structures and processes are embedded. Theories which deemphasize the role of conceptual structures tend to hypothesize relatively powerful processes (see Figure 2a), whereas theories which hypothesize the existence of powerful structures need only relatively weak content specific processes (see Figure 2b). Our perspective goes one step further, assuming not only the existence of powerful structures but also that the processes contribute to the meaning of the underlying ideas (see Figure 2c).



(a)
Weak Structures
Powerful Processes

(b)
Strong Structures
Weak Processes

(c)
Conceptual
Models

Figure 2

The APS project emphasizes the "structured wholeness" of conceptual models, rather than assuming that these systems are built up by linking together relatively discrete condition-action pairs.

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STRATEGIES AND ERRORS IN GENERALISED ARITHMETIC

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ABSTRACT: L'examen des stratégies et erreurs des enfants en algèbre élémentaire a commencé par l'analyse des données du test d'algèbre des Concepts in Secondary Mathematics and Science (CSMS), afin d'identifier les erreurs les plus courantes et d'en proposer les origines vraisemblables. A la suite de cette analyse on a choisi dix questions dont les fausses réponses, selon l'analyse des CSMS, paraissaient être le résultat de la manipulation des lettres en algèbre élémentaire, auxquelles les élèves ont donné une valeur, ou qu'ils ont tout à fait méconnues, ou qu'ils ont considérées comme objets. On a dévelopé un programme d'interviews individuelles à la base de formes parallèles de ces questions pour permettre l'examen des erreurs. On a donc choisi 55 enfants âgés de 13 à 16 ans au niveau moyen en mathématiques dans cinq écoles de la banlieue de Londres. L'analyse des réponses à l'interview indique deux autres domaines difficiles en algèbre élémentaire au delà de celui de l'interprétation des lettres. Il s'agit de la manière dont l'enfant essaie de résoudre le problème, et sa façon d'en codifier la réponse. Ces résultats sont présentés sous forme de données en entrée-traitment-sortie.

The Strategies and Errors in Secondary Mathematics (SESM) project is a project funded by the Social Science Research Council and based at Chelsea College. This project follows on from the work of the Mathematics section of the Concepts in Secondary Mathematics and Science (CSMS) project, which was also based at Chelsea College, and aims to investigate particular widespread errors in mathematics which were identified by the latter project.

In the case of generalised arithmetic, these errors were suggested to arise largely as the result of the child's interpretation of letters (Küchemann, 1978, 1981a, b). The 'strategies and errors' investigation thus began with an analysis of the CSMS Algebra test items and data in order to identify those items to which particular wrong answers were occurring with high frequency (approximately thirty per cent or more). As a result of this, ten CSMS test items were selected for study (see Figure 1 for examples), the erroneous answers to which could be interpreted as being due to the child's handling algebra not as generalised arithmetic, but rather by ignoring the letters, by substituting alphabetic or other values for the letters, or by treating the letters as objects (Küchemann, op.cit.). At the same time, the viewpoint that many children may not be operating in terms of the system of 'school mathematics', but may rather be relying upon their own intuitive 'child-methods' (Booth, 1981; Erlwanger, 1975; Ginsburg, 1977; Hart, 1981), suggested that the methods used by children, as well as the meaning ascribed to letters in generalised arithmetic, may be contributing to the observed errors. Consequently it was thought desirable to investigate these various possibilities.

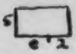
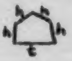
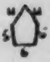
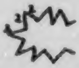
	CSMS Item (abridged)	'Error' Answer	Percentage Giving Answer (13 yr.olds)
1.	Area of: 	5e2, e10, 10e, e+10 10, 7	41.7 9.5
2.	Perimeter 	hhhht, 4ht, 5ht	26.8
3.	Perimeter 	2u556, 2u16	46.3
4.	Perimeter: (n sides of length 2) 	n2 32 to 42	14.5 25.4
5.	Add 4 onto 3n	3n4, 7n 7, 12	44.7 17.3
6.	Multiply by 4: n+5	4n5, n45 n+20, n+9 20, 9	11.5 38.9 16.4
7.	Simplify if you can: 2a+5b	7ab, 8ab 7	45.3 5.5

Figure 1.

When a child is presented with a particular problem or item and responds to it, any error observed in his or her final answer may have entered the child-item system at one of several points. The child may have misinterpreted either the elements of the item or what is required by the item, or he may have used an incorrect method or approach in solving the problem, or he may have encoded the result incorrectly. (There is also, of course, the possibility of any combination or interaction of such errors). For example, consider the child who gives an answer of the type 5e2 or e10 or 10e to question 1 in Figure 1. This error may have arisen because the child:

a) effectively interpreted the letter as a 'thing' which could be merely collected up with the numbers (input error);

b) realised that 'e' represented a number expressing part of the length of the base, but thought that area meant multiplying everything together (process error);

c) did not know how to interpret the letter or did not know how to operate with it and so performed the numerical calculation and 'put down' the letter afterwards (input/process error);

or d) interpreted the letter correctly, applied the correct method, knew to add the 'e' and '2' before multiplying by 5, but recorded the 'answer' to e added to 2 as 'e2' (or '2e') (output error).

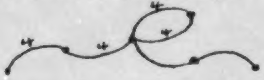
(Of course, there may be other possibilities as well). The consideration of an input-process-output model of this type permits a clearer picture to be obtained concerning the point(s) at which the child's understanding of the problem breaks down. An interview schedule designed to separate out these components of the child's problem-solving process, and based on the error-analysis approach elucidated by Newman (Casey, 1978; Clements, 1980; Newman, 1977) was thus developed for use in individual interviews.

In order to select children for interview, the CSMS Algebra test was given to a total of 201 children aged 13 to 16 years in the 'middle ability' mathematics groups of five schools in the outer London area. The tests were administered by the class teachers in normal mathematics periods, and were given four to six weeks prior to the commencement of the interview programme. A total of forty-eight children from this sample was selected for individual interview on the basis of their performance on the CSMS Algebra test and the criteria concerning level of letter-interpretation, previously outlined (letter ignored, evaluated or treated as object), and a set of questions of 'parallel form' to the ten items under study was drawn up to form the basis of the interviews. As a check on the representativeness of the test-sample from which the interviewees were chosen, CSMS Algebra levels (CSMS Mathematics Team, 1981) were allocated to all the children tested, and the distribution of levels obtained was compared with that derived from the CSMS large-sample (N=2820) data. The closeness of fit observed between the two distributions was consistent with both SESM and CSMS samples being regarded as representing the same population. During each interview, which lasted approximately thirty minutes, the child was asked to explain how he or she would interpret, solve, and record the answer to each question presented. The interviews were tape-recorded and subsequently transcribed by the interviewer. Seventeen of the 48 children interviewed were subsequently re-interviewed on a second interview schedule (including some repeated items to check for consistency of error and approach) six months after the first round of interviews, and a further seven children new to the project were also interviewed at this second stage. The purpose of the second-stage interviews was to investigate specific hypotheses concerning areas of misunderstanding which had been formed as the result of the round-one interviews.

From the analysis of the first set of interviews it became apparent that while children do show some confusion over the meaning of letters, this was not the only source of error. In terms of the input-process-output model, such 'interpretation' errors may be regarded as 'input errors', since they relate to the way in which the child 'receives' the original data. By means of the interview schedule referred to above, however, it was also possible to identify two other sources of error, one relating to the processing of the problem once interpreted, and the second to the way in which the child encodes the answer obtained. During interviews on the 'perimeter' items (items 2 to 4 in Figure 1), for example, it became apparent that part of the difficulty which children experienced with item 4 was due to the need to change from an additive to a multiplicative method for that item. Children who solved the earlier perimeter items by adding the sides and who attempted to extend this method to item 4 were faced with the problem of adding 'n' lots of 2, a problem which was resolved by 24 of the 55 children interviewed (44%) by assuming a particular number of sides, while a further 11 children (20%) stated that they would need to know how many sides to add before they could proceed. In addition, it became apparent that an ability to verbally describe a method does not necessarily entail a recognition of the fact that that method can be symbolised mathematically (or perhaps that such symbolisation may be an appropriate thing to do). It seems that children may have ways of doing things which they do not formalise (see also Brown, 1981; Brown and Küchemann, 1976, 1977); if such methods are not formally stated, the chances of the child's producing an algebraic representation of those methods are understandably slight. Certainly, where children were observed to give an appropriate formal statement of the method (e.g. 'multiply the number of sides by the length of each side'), there appeared to be no difficulty in giving the correct algebraic statement.

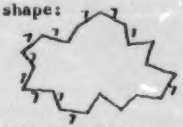
These notions were explored in the second round of interviews, in which a series of items of the kind shown in Figure 2 was presented.

1. A spaceship travels in 'stages' which are all the same distance long:



If each 'stage' is 4 light-years long, what could you write for how far the spaceship goes in 97 'stages'?

2. What could you write for the perimeter of this shape:



All the sides are of length 7.
There are 19 sides altogether.

Figure 2

Relatively large numerical values were used in these items in order to focus the child's attention on the operation to be used rather than the answer, and in order to encourage the use of multiplication, so that no change in operation would be necessary when the algebraic test item was introduced. Under these circumstances, 20 of the 23 children (87%) interviewed on these items gave the correct algebraic statement, as compared with 11% (6 out of 55) on the first round of interviews.

The possibility of a further contributing factor to this change in performance was, however, also suggested by these second-round interviews. This factor relates to the possibility of a communication problem between child and teacher which may partly underlie the observed difficulty in generalised arithmetic. For many children an algebraic expression is not regarded as a legitimate 'answer'; consequently these children are either reluctant to give such an answer, assuming that something else must have been intended, or they will derive a correct algebraic statement but then use various substitution devices in order to obtain from it a numerical answer. In this case, the problem lies not so much in attempting to record an inappropriate method, nor in not having a formal model of a method which can be mathematically symbolised, but rather in the legitimacy of recording any statement at all in its (to the child) 'incomplete' form. The notion of the child's inability to accept lack of closure has been discussed in general terms by Collis (1972) and Lunzer (1976), and in the specific context of algebra by Matz (1979) and Davis, Jockush and McKnight (1978). It appeared from the SESM interviews, however, to be more the case that the children considered that a 'closed' answer was required, rather than that they could not conceive of any other possibility. By 'allowing' the children to leave their 'answers' to the items in Figure 2 in the unclosed form 97 x 4 and 19 x 7, the way was left clear for them to feel free to do the same thing in the case of the algebraic test items. This factor may well contribute to the apparent ease with which the children handled the latter items.

This apparent desire to give a 'final answer' may also account in part for one of the 'output' errors observed. Thus 9 out of 20 children (45%) interviewed on item 1(Figure 1) in the second round of interviews correctly stated that the area of the rectangle could be found by adding '2' and 'e' and then multiplying the answer by '5', but then went on to state that 'e plus 2' could be recorded as 'e2'. This encoding error was observed consistently across several items to which it was applicable. In addition, the responses given by children to this item (and also to item 6 in Figure 1) confirmed previous observations (e.g. Kieran,1979) concerning children's reluctance to use brackets.

While the data so far gathered are not extensive, being based on a sample of 55 children in total, the consistency (across items) and stability (across time) of the observed errors and the underlying sources for them which children reveal, would seem to indicate that the 'problems' in generalised arithmetic which the study has outlined may be fairly firmly established in the secondary-school population at large. If this is so, then the implications of such misconceptions for further study in algebra perhaps need no elaboration. Further investigation to discover ways in which the child may be guided to adjust his way of viewing generalised arithmetic problems is now being undertaken.

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11.

EARLY ADOLESCENTS' REASONING WITH UNKNOWNS

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La résolution de nombreux problèmes mathématiques nécessite de chercher une grandeur "incunnue" pour laquelle l'énoncé ne donne aucunne information directe. Pour étudier le processus par lequel se fait la prise de conscience d'une telle inconnue, nous avons posé une série de huit problèmes arithmétiques à 130 élèves âgés de douze et quatorze ans. Les résultats montrent que la plupart des élèves a réussi à trouver quelles étaient les inconnues, certains à partir de la réponse qui était donnée (résultat le plus fréquent), d'autres par essais et erreurs (second résultat en fréquence), d'autres quelquefois en devinant la reponse, ou d'autres, rarement, en utilisant une équation. La presence de schémas illustrant les énoncés a peu d'influence. Il n'y a guère de différence entre garçons et filles, et entre élèves de douze ou quatorze ans.

Arithmetic and algebra word problems have a reputation of presenting great difficulties to many students. A substantial effort in mathematics education research has been devoted to studies of word problem solving (see, for instance, Goldin and McClintock, 1979; Lesh, Mierkiewicz, and Kantowsky, 1979). The key feature of the word problems with which we are concerned is that they require the solver to (1) conceptualize one or more unknowns and (2) interpret the problem conditions as operations on the unknown(s) yielding a specified result.

Our research has been carried out in the theoretical framework of reasoning patterns (Karplus, 1977; Karplus, 1981; Karplus, Pulos, and Stage, 1981), which are identifiable, reproducible thought processes directed at a particular type of task. The use of a reasoning pattern, such as reasoning with unknowns, is postulated to show continuous development from successful application at a concrete level (familiar actions and objects, observable properties, simple correspondences) to later success at a formal level (complex relationships, intangible properties, hypotheses contrary to fact, transformations). Development need not be unidimensional and need not pass through levels that can be characterized in a well-defined way applicable to all reasoning patterns.

Past research on problem solving has concentrated on the effect of certain task variables on subjects' success rates in solving problems (Goldin and

McClintock, 1979) rather than on a description of the procedures that students actually use to succeed or fail in solving the problem. We have therefore pursued the latter goal, with early adolescent students who had studied algebra only very slightly if at all. The tasks were designed to answer the following questions:

1. What forms of reasoning with or without unknowns do early adolescents actually apply on selected word problems?
2. What are the frequency distributions of the various patterns of reasoning with unknowns?
3. How is reasoning affected by the complexity of the problem?
4. How is reasoning affected by the presence of a diagrammatic representation of the problem, by the subjects' sex, and by their grade level?
5. How consistently does a subject employ a particular reasoning pattern?

THE NUMBER PUZZLES

The word problems we used were of the form,"I am thinking of a number. I add 12 to my number and then multiply by 6. I get 90. What is my number?" These abstract problems avoided misunderstandings that might be generated by a concrete context and by key words (more than, twice as much as..., altogether) that merely imply mathematical operations.

Each Number Puzzle was administered in an interview by means of a card that was placed before the student and was also read aloud. Each student was asked for the solution and then for the solution procedure. The Number Puzzle Task consisted of eight puzzles, all of the same form, with whole number unknowns. The specific data we used are listed in Table 1 in the order of presentation. The eight puzzles formed three groups: one step puzzles 1, 2, and 3; two step puzzles 4, 5, and 6; and "loop" puzzles 7 and 8 in which the unknown appeared in two places. For about half of the subjects the verbal presentation on the card was accompanied by a flowchart diagram like the one below for Puzzle 4.

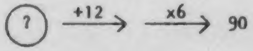

Table 1. Number Puzzles in Equation Form

Puzzle	Equation	Puzzle	Equation	Puzzle	Equation
1	$N \times 5 = 65$	4	$(N+12) \times 6 = 90$	7	$(N \times 4) - 21 = N$
2	$N - 12 = 9$	5	$(N \times 4) - 12 = 32$	8	$(N-16) \times 3 = N$
3	$N \div 8 = 2$	6	$(N \div 8) + 11 = 16$		

THE SUBJECTS

Our 130 subjects consisted of approximately equal numbers of sixth and eighth grade boys and girls from a middle class suburban school with an ethnically mixed population performing near the average of standardized mathematics tests. They were selected randomly from an enrollment of about 150 in each grade.

RESPONSE CATEGORIES

For the purposes of this preliminary report, we classified the students' procedures on each puzzle into the following seven categories:

Category N -- item omitted or subject made no progress;
Category C -- computing inappropriately (e.g. 12-9=3 on Puzzle 2);
Category G -- guessing the answer after one or several unrelated trials;
Category T -- trial-and-error cycles progressing systematically;
Category B1-- working one step backwards from the stated result;
Category B2--'working two steps backwards from the stated result;
Category E -- forming an equation and solving it.

We interpreted the use of procedures that fell into Categories G, T, B1, B2, or E as evidence that the subject was reasoning with an unknown, though not necessarily correctly. We judge that the procedures increase in merit in the order listed.

RESULTS AND DISCUSSION

We shall now proceed to answer the five research questions in order. The forms of reasoning used by our subjects are reflected in the response categories, which were derived from the data.

Frequency distributions. Our principal results, the frequency distributions among the response categories, are presented in Table 2. The percentages of correct answers are included on the last line. It can be seen that most of the subjects did reason with an unknown on most of the puzzles, but that there was considerable variation in the reasoning patterns employed.

Table 2. Frequency Distributions of Response Categories (percent)

	Puzzle Number							
Category	1	2	3	4	5	6	7	8
N	0.0	0.8	0.0	1.5	2.3	2.3	3.8	9.2
C	1.5	15.4	19.2	6.2	2.3	7.7	3.8	6.9
G	11.5	5.4	12.3	10.0	12.3	14.6	33.8	24.6
T	42.3	3.1	0.0	16.2	29.2	20.0	48.5	50.0
B1	42.3	71.5	67.7	3.1	0.8	3.8	0.0	0.0
B2	0.0	0.0	0.0	60.8	49.2	48.5	0.0	0.0
E	2.3	3.8	0.8	2.3	3.8	3.1	10.0	9.2
% correct	96.9	83.1	80.0	66.9	88.5	80.8	72.3	61.5

As might be expected, the frequencies of working one or two steps backwards were high for the first and second groups of three puzzles respectively, and such reasoning did not occur at all on Puzzles 7 and 8. These were solved most frequently by trial-and-error approaches or guessing. From a surprisingly high level of trial-and-error tactics on Puzzle 1 we concluded that many students may actually have been working backwards but used trial-and-error to carry out the required division operation.

The two apparently very simple Puzzles 2 and 3 showed a surprisingly high frequency of inappropriate computation (Category C). Most of the computations took the form 12-9=3 for Puzzle 2 and 8-2=4 for Puzzle 3, arithmetic applied to the given numbers and operations to obtain a natural number answer without regard to the structure of the problem. When students checked these answers, they again used their inappropriate procedures rather than rereading the puzzle statement, and therefore did not recognize their mistakes.

Guessing -- usually successful -- occurred with a frequency of about 10% on the first six puzzles, and a frequency of about 30% on the last two. This category may include responses from some individuals who actually worked backwards or used a systematic trial-and-error approach but did not articulate this procedure in the interview.

In spite of the high frequency of working backward strategies on Puzzle 4, the percentage correct was lower than for Puzzles 5 and 6. Inspection of the students' calculations showed that many worked backwards incorrectly in that they did not make use of the distributive principle implied in the puzzle statement. That is, they solved the puzzles represented by the equations 6N+12 = 90 or N+(6x12) = 90 rather than (N+12)x6 = 90.

Puzzle difficulty. None of the information we have provided so far gives a clear indication of the relative difficulty of the puzzles since students were encouraged to work until they were satisfied with the answer. We believe that a measure of the time required, such as the time for two-thirds of the subjects to complete a puzzle, is a good indicator of difficulty. For Puzzles 1, 2, and 3 this time was one minute or less, for Puzzles 5 and 6 it was two minutes, for Puzzles 4 and 7 it was three minutes, and for Puzzle 8 it was four minutes. The progressive increase in working time did follow the puzzles' complexity as we have described it previously. The additional time required for Puzzle 4 compared to the other two-step Puzzles 5 and 6, and by Puzzle 8 compared to the other loop Puzzle 7, reflect the need for using the distributive principle.

Format, sex, and grade. We were surprised to find that the reasoning patterns were applied with very similar frequency distributions, regardless of the presence of a diagram, whether the subject was a boy or girl, or whether sixth graders were compared with eighth graders. The only statistically significant difference occurred in the use of equations by 14 eighth graders compared to one sixth grader.

Consistency of reasoning. Since the subjects' goal was to find the answer to the puzzles by any method, there was little external incentive for their using the same reasoning pattern consistently. To describe our findings, we have defined consistency as the use of a particular approach five or more times on the eight puzzles. With this criterion, 4% of the subjects consistently failed to reason with an unknown, 15% consistently guessed or used trial-and-error, and 43% consistently worked backwards or stated equations. The remaining 38% of the students used the various approaches with less consistently.

CONCLUSIONS

The students participating in the Number Puzzle interviews reasoned with unknowns to a very great extent and usually found the correct answer. The most frequent reasoning pattern was to work backwards from the given result, with trial-and-error the second most frequent strategy. Errors arose from incorrect manipulations, most commonly in two-step puzzles where the distributive principle had to be applied.

The Number Puzzles were constructed as a particularly simple type of word problem that required reasoning with one unknown but little translation from words to relationships. Our results suggest that the difficulty many students have in solving more general word problems with one unknown is due more to their incorrect translation of the verbal clues into mathematical relationships than to their inability to conceptualize one unknown.

Interpreted developmentally, our results suggest that guessing or trial-and-error strategies applied to our puzzles with small whole-number unknowns are concrete level applications of the reasoning pattern, as is working backwards one step. These approaches require only a step-by-step procedure or a single negation. Working backwards two steps and using an equation successfully are formal level applications that require proper sequencing or representation of two operations. While our data on the use of the latter strategies showed some advance between grades six and eight, a more effective test of development of reasoning with unknowns would require a task with an explicit need for reasoning backwards. Number puzzles with non-integer solutions could be used for this purpose.

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UTILISATION DES DECIMAUX DANS DES PROBLEMES D'APPROXIMATION

par R. DOUADY et M.J. PERRIN

Our work assumes the following hypothesis : a mathematical concept has its meaning in the way one uses it, in the problems which can be solved with it.

Our problem : to construct didactical sequences on decimal numbers - with an arbitrary precision - plays an essential role.
To study the implications of this viewpoint on the behaviour of the pupils, on their choice of a strategy in problems having no decimal solution but a real one.

The construction of such didactical sequences involves the choice of a semantical frame in which the existence of non rational numbers - numbers which do not appear as result of operations +, -,x,: - takes signification.

In this purpose, we have chosen to work in geometry and ask problems of existence of figures or points. Such problems admit a numerical or algebric expression, and traduction into this setting is needed by the children for their solution.

These problems and the rule imposed to the children will be different in learning sequences, in written knowledge tests, in individual interviews.

We shall analyse the procedures of the children in clinical interview according to three aspects
- *meaning of the existence of a solution*
- *searching of variation law, role of the unknown quantities*
- *role and working of decimal numbers.*

The interaction of these aspects will be considered. We shall try to compare these procedures with those used in learning sequences.

Notre travail prend son origine dans l'hypothèse suivante : une notion mathématique prend son sens dans l'usage qui en est fait, dans les problèmes qu'elle permet de résoudre.

NOTRE PROBLEME

Construire des séquences didactiques sur les nombres décimaux dans lesquelles leur propriété d'approcher les nombres réels avec une précision arbitrairement grande joue un rôle essentiel.

Etudier l'incidence de la prise en compte de ce point de vue sur les conduites des élèves, sur leurs choix de stratégie pour résoudre des problèmes n'ayant pas de solution décimale mais ayant une solution réelle.

SITUATIONS-PROBLEMES POUR LES ELEVES

La construction de ces séquences didactiques pose le problème du choix du cadre sémantique propre à conférer de la signification à l'existence des nombres réels non rationnels, c'est-à-dire des nombres qui n'apparaissent pas comme résultat, à partir des entiers, d'opérations (+,-,x,:).

Nous avons choisi, pour ce faire, de nous placer en géométrie et de poser des problèmes d'existence de figures ou de points répondant à certaines contraintes.

Exemples : 1) Parmi les rectangles d'aire fixée, existe-t-il un carré ?

2) Parmi les rectangles de périmètre donné, en existe-t-il un d'aire maximum ?

3) Existe-t-il un rectangle dont le périmètre et l'aire soient donnés à l'avance ?

4) Etant donnés 2 points A et B sur une cercle Γ, peut-on choisir un troisième point C sur Γ

- de manière que l'aire du triangle ABC soit maximum
- de manière que l'aire du triangle ABC soit donnée à l'avance.

. La situation didactique relative au problème 1 proposée à des élèves de 8-9 ans a été décrite (R.Douady 1980). Elle a donné lieu à l'explicitation des nombres décimaux et de leur écriture sous forme standard.

. Les problèmes 2 et 3 ont fait l'objet avec ces mêmes élèves 2 ans plus tard de séquences en classe sous diverses formes (prob.2 en Nov. Déc.79 ; prob.3 en janv.80). Au préalable, nous avions fait, pour le problème 3, une analyse mathématique et un recensement des méthodes que les élèves étaient susceptibles de mettre en oeuvre pour le résoudre. Nous pensions que le choix de la méthode dépendrait du sens accordé à l'existence d'une solution, mais aussi que ce sens pouvait évoluer en fonction des informations obtenues, suivant qu'il était possible d'exhiber ou non la solution. Si on pose le problème plus général de reconnaître parmi tous les couples (x,y) ceux pour lesquels il existe un rectangle de demi-périmètre X et d'aire Y et de prouver la non existence dans certains cas, on est conduit à se poser le problème 2.

Les consignes données aux élèves ont été choisies, après cette analyse, de manière qu'ils aient à expliquer et justifier leurs choix et leurs décisions auprès d'un contradicteur.

Le problème 3 a été posé 6 mois après les séquences d'apprentissage en juin 80 à certains des élèves dans un cas où il y avait une solution irrationnelle, dans un cas où il n'y avait pas de solution.

Le problème 4 a été posé en entretien individuel en juin 80.

PROCEDURES

Nous analyserons les procédures des enfants pour résoudre les problèmes 3 et 4 lors des entretiens selon trois points de vue :

- signification de l'existence d'une solution
- recherche de lois de variation, rôle des inconnues
- rôle et fonctionnement des décimaux

et les interactions entre ces trois aspects.

Les procédures se classent en trois catégories.

1) PROCEDURES EMPIRIQUES

Pb 3 : On choisit a,b tels que $a+b = 41$ et $a \times b$ proche de 402. Puis on modifie de façon à se rapprocher de 402 sans trop s'éloigner de 41.

Pb 4 : On choisit quelques points à l'oeil sur le cercle et on calcule l'aire correspondante. "On voit" que le point le plus haut donne l'aire la plus grande

2) PROCEDURES EXPERIMENTALES SOUTENUES PAR UN ARGUMENT THEORIQUE

Pb 3 : a) On garde a+b fixe égal à 41, on fait varier (a,b) de manière que axb se rapproche de 402. Le sens de variation de l'aire est utilisé pour orienter les choix de a et b.

b) On garde a x b fixe égal à 402 et on fait varier (a,b) par une procédure analogue à la précédente.

c) On alterne entre les procédures a) et b).

Pb 4 : On recherche le sens de variation de l'aire en fonction de la position de C sur Γ : à 2 points symétriques par rapport au diamètre vertical correspondent des aires égales. L'aire croit puis décroit quand C se déplace de gauche à droite et prend la valeur maximum quand C est sur l'axe de symétrie.

3) PROCEDURE RESULTANT D'UNE PREVISION THEORIQUE

Pb 3 : - on répond à l'existence par référence à l'aire du carré (a + b = 41 ou a + b = 39)

- dans le cas où il y a une solution on en cherche une valeur numérique approchée en gardant a + b = 41.

Pb 4 : a) On recherche la plus grande hauteur possible. Ce choix est justifié soit par l'argument suivant : la base AB est fixe, la plus grande aire correspond à la plus grande hauteur.

soit par des calculs d'aires correspondant à des points voisins.

b) On recherche un triangle isocèle pour rendre les 2 autres côtés du triangle "le plus grand possible en même temps".

c) On choisit de prendre un diamètre comme côté parce que c'est le plus long qu'on peut tracer.

d) On choisit une hauteur de même longueur que la base.

Toutes ne mènent pas au succès, d) traduit une volonté d'utiliser le problème 2 traité en classe.

15.

Nous ne pouvons pas détailler ici les procédures du problème 4 sous sa deuxième forme. Nous pouvons dire que les élèves concluent ou non à l'existence de solution selon le sens accordé à ce mot, à peu près indépendemment de la procédure employée, sauf ceux qui se réfèrent à l'aire du carré pour le problème 3 ($a + b = 39$ $a \times b = 402$) dans le cas où il n'y a pas de solution. Tous utilisent les décimaux pour chercher une solution approchée en cherchant à réduire l'intervalle d'incertitude. Cette réduction se fait souvent en recherchant le sens de variation de l'aire (Pb 3, en fonction de a,b et en prenant pour règle que l'aire croît quand $|a - b|$ décroît, Pb 4 en fonction de la hauteur de C). Cette règle peut résulter d'un constat après calculs ou d'une représentation graphique des couples (a,b) pour le Pb 3 en indiquant à côté de chaque point la valeur de $a \times b$.

On constate à cette occasion que, dans l'ensemble, les élèves ont une bonne maîtrise des opérations et de l'ordre sur les décimaux.

. Nous avons aussi fait passer un test écrit de connaissance sur l'ordre des décimaux dans 3 CM2. Nous les présenterons et les commenterons.

. Nous avons proposé en séquence de classe les problèmes 1,2,3 à des élèves de 6è (11 ans) ayant déjà une certaine conception des décimaux, laquelle cependant ne leur permettait pas de donner une solution approchée à un problème. Nous avons étudié l'évolution de leurs conceptions et de leurs procédures au cours de ces problèmes. Nous envisageons de les interroger en entretien individuel sur les problèmes 3 et 4. Nous comparerons leurs procédures et leurs convictions à celles des élèves de CM2. En particulier, nous nous intéresserons, dans la recherche d'un triangle d'aire donnée (Pb 4) au rôle joué par l'utilisation éventuelle d'une équation algébrique.

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PRE-ALGEBRAIC NOTIONS AMONG 12- AND 13-YEAR OLDS

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Cette communication décrit une étude examinant la conception qu'ont déjà les élèves, avant tout enseignement formel de l'algèbre, au sujet de l'égalité, la notion de variable et la résolution d'équations. Dix étudiants montréalais en première année du secondaire ont été choisis comme sujets (agés de $12\frac{1}{2}$ à $13\frac{1}{2}$ ans). L'entrevue clinique (sans aucun enseignement) a été la méthodologie utilisée dans cette recherche; les entrevues individuelles ont été enregistré sur cassettes. Les résultats indiquent que les élèves au seuil du secondaire semblent connaître et semble savoir faire un peu d'algèbre à un niveau intuitif et non-formel. En se basant sur leur familiarité avec des équations très simples et les opérations inverses, dont ils avaient fait l'experience au primaire, ils peuvent étendre ces connaissances à la résolution d'équations plus élaborées. Deux types de questions ont été employé pour sonder la conception que pouvait avoir les élèves de la notion de variable: des formes indéterminées telles que $3a$, $a + 3$, $3a + 5a$; ainsi que différentes équations du premier degré.

This paper describes the results of a study which investigated some notions which children, who have not yet been taught any formal algebra, bring with them when they begin their high school algebra course. The study was designed to research not only what the students knew about equality, variable, and the solving of equations, and what kinds of understanding they had, but also those areas where they would try (or not try) to extend their existing knowledge and how they would do so.

METHOD

The research reported herein is the first part of a larger teaching experiment on the learning of algebra. This preliminary phase was conducted in November and December, 1980. The subjects were 10 Montreal students of average mathematical ability in their first year of secondary school (a school fed by six different elementary schools). They ranged in age from $12\frac{1}{2}$ to $13\frac{1}{2}$ years. The mathematics course which these students were following in school consisted primarily in a review of their arithmetic skills; algebra is not taught until the second year of secondary school. The research methodology used was the clinical method (no teaching involved); the individual interviews were audio-taped.

The main topics probed in this study, along with the rationale for each, are summarized in TABLE 1. Questions dealing with each of these areas were designed in advance; however, interesting, unusual, or ambiguous answers by subjects were pursued further for purposes of clarification. Subjects were asked to think aloud; they could also use paper and pencil whenever they wished. Thus, data consisted of the verbal protocols and the written work of the subjects.

TABLE 1

Topics Probed

TOPIC	RATIONALE
A. In domain of arithmetic	
1. Notions of equality and uses of equal sign.	1. Do students see equal sign as an operator symbol or as a symbol for equivalence (Kieran, 1980)? Hyp.: Student's view of equality may influence solving strategies used.
2. Computational skill.	2. Do large numbers affect their computational skill? Hyp.: Algebraic equations with large coefficients may seem more difficult than those with small coefficients.
3. Knowledge of inverses, identity elements, operations with negative integers, conventional order of operations, use of brackets.	3. What awarenesses do they have of the relationship between addition and subtraction, multiplication and division? What are their intuitive ideas on operating with strings of arithmetic operations?
B. In domain of algebra	
4. Notions of algebraic equations. a) their own examples b) their reactions to examples proposed by researcher (various combinations of occurrences of unknown and coefficients)	4. What is their previous exposure to equations? How will they react to types they have never seen before? What will they try to do with them?
5. Interpretation of letters.	5. Do they conserve equation (Wagner, 1977)? Does their interpretation vary with the location of the letter in equation?
6. Notions of indeterminate expressions (eg, $a + 3$, $3a$, etc.).	6. What kinds of meanings do students give to such expressions?
7. Processes used in solving simple equations of the forms: $ax = b$, $x + a = b$, $ax - b = c$, $ax + b = c$, $x/a + b = c$.	7. Will the same examples with large coefficients be more difficult? Hyp.: Subjects may use an undoing strategy for equations with large coefficients and a plugging-in strategy for equations with small coefficients.

RESULTS*

The equal sign. Subjects confirmed findings of previous studies (Herscovics & Kieran, 1980; Kieran, 1980): 8 of the 10 subjects explained the meaning of the equal sign in terms of an operation on the left side and the answer on the right, and gave examples, such as, $6 + 2 = 8$ and $20 \times 5 = 100$. The other 2 subjects stated that the equal sign meant "equal amounts on each side" and "both things on each side of it are the same -- the total of it is the same" and provided as examples, $6/5 = 1\ 1/5$ and $3 \times 6 = 2 \times 9$ respectively.

When subjects were then asked what they thought of the use of the equal sign in the examples, $5 = 3 + 2$, $8 + 2 = 5 \times 2$, $7 = 7$, they readily accepted the first two uses. However, 5 of the 10 subjects thought that the written form of $7 = 7$ made no sense: "You don't need to work it out; you got it already. Everybody knows that"; "Why write that; we might see $5 + 2 = 7$"; "7 is equal to 7, I don't know why you have to put the equal sign"; "It's the same, although it doesn't mean anything to me".

Notion of an algebraic equation. Though none of the subjects knew what the term "equation" meant, they had all previously seen some examples of them, such as $n + 4 = 20$, $6 \times b = 72$. In fact, when asked what they did with "things" like these, they solved them by undoing, that is, $20 - 4$ and $72 \div 6$. When asked if they had ever seen anything like $3x + 4 = 10$, they all said that they never had and that they did not know what the 3x meant. They were then asked what they thought of $3 \times n + 5 = 17$. Though 5 subjects said they had seen this type before, only 4 of the 5 had a correct idea of what to do with it: "17 - 5 and divide by 3"; or "3 times what plus 5 equals 17 -- 3 times 4 is 12 plus 5 is 17". The remaining subject suggested,"Add 3 and 5, then divide 17 by 8". The other five subjects had never seen two operations on the same side of the equals. They nevertheless attempted to handle it by either adding 3 and 5, then subtracting 8 from 17; or multiplying 3 and 5, indicating then that n would be 2, because 15 and 2 are 17.

None of the subjects had ever seen $x + 5 + 7 = 4 - 3 + 15$ before, yet they all spontaneously solved it by combining the numerical terms (9 of the 10) in a left-to-right order (thus confirming the findings of a previous study (Kieran, 1979)); the tenth subject used a right-to-left procedure. It is obvious that

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[*] Due to space constraints, both the results and their discussion have had to be summarized considerably.

none of them would have solved it correctly if the first operation had been a multiplication, i.e., $n \times 5 + 7 = 4 - 3 + 15$, for none of these subjects had yet been taught the conventional order of operations. However, the point to be made is that, as long as there is only one occurrence of the unknown, and as long as all of the operations are symbolized explicitly, even novice students will probably try to solve such equations, usually by combining the numerical terms on each side and then undoing as a last step.

When subjects were presented with equations containing two occurrences of the unknown, which 8 of them stated they had never seen before, many of them seemed unable to extend their existing knowledge to cover such situations. With the equation $x + 2 = 2 + x$, which required no combining or undoing, only 4 of the 10 subjects seemed able to use their notions of arithmetic equality to cover this example: "You could put any old number, as long as it gives you the same answer"; "they're both 2 and they're both equal, so they should both be the same"; "x can be zero or any number, because the two of them are the same number, 2 and 2"; and so on. When asked about $3 + a + 2 = 10 + a$, the same 4 subjects stated that the two a's would have to be different; the other subjects could make no sense of an equation containing two occurrences of the unknown.

Interpretation of letters. When Wagner (1977) presented the two equations, $7 \times w + 22 = 109$, and $7 \times n + 22 = 109$, to 14 twelve-year-olds who had not had any formal algebra, and asked, "If you were to figure out what w and n would be to make these statements true, which would be larger, w or n?", her results indicated that only 5 of the 14 subjects said that w and n would have to be the same. This task was repeated with the 10 subjects in this study; however, the wording of the question was changed to, "Are the solutions to these two equations the same or different?" All 10 subjects replied that both n and w were the same and, in order to justify their response, provided reasons in terms of the other identical features of the equations.

When subjects were asked the meaning of the letter in $5 + a = 12$ and $n \times 3 = 15$, they responded in terms of its value. However, all but one (who guessed) did not know the meaning of the letter in $2c + 15 = 29$. They had never seen $2 \times c$ written as $2c$. Four of these 9 subjects were then told of the convention in order to see if they could solve equations which were written in concatenated form. As will be seen later, though concatenation may have been a factor in the solving of two-operation equations, it obviously had no effect on any of the subjects in their dealing with single-operation equations (eg, $6a = 18$).

Indeterminate expressions. In order to probe further into their concepts of variable and unknown, subjects were asked the meaning of $3a$, $a + 3$, $3a + 5a$. For the expression, $a + 3$, which did not carry the concatenation problem with it, 7 subjects could not assign any meaning, because they could not find the value of a. "Meaning" for them was only defined in terms of finding some answer, some value for a: "I can't tell what a is until I know what this equals"; "I don't understand"; "if there was an equal sign -- "; "a has no value". The other two examples, $3a$ and $3a + 5a$, proved meaningless not only to 5 of the 6 who did not know about concatenation, but also to 2 of those who did, "If we had the answer, like $3a = 30$, we could do it, but we don't know".

Novices' inability to assign any general meaning to indeterminate expressions would seem to indicate that early instruction in the gathering of like terms, for example, "simplify $3a + 5a$", ought perhaps to be postponed until students see such expressions within the context of equations, that is, $3a + 5a = 16$. The presence of the equal sign would seem to be a necessary ingredient in the early stages of a high school algebra course, and perhaps should also be considered in the introduction of polynomials which traditionally have been treated as indeterminate expressions and have thereby caused students a fair amount of difficulty.

The solving of equations. The last section of this study dealt with the strategies used in the solving of the nine equations shown in TABLE 2. For equations of the forms $ax = b$ and $x + a = b$, the most common solving procedure used was "undoing", that is, performing the inverse operation, such as, $b \div a$ and $b - a$. The only exception was the equation $6a = 18$ which 7 subjects solved by "number facts", that is, immediately replacing the unknown by the correct value and reading it in the order in which it was written, "6 times 3 is 18". Familiarity with the multiplication facts up to and including 10 is considered to be the reason for employing this particular strategy with this equation.

For equations of the forms $ax + b = c$ and $ax - b = c$ which were solved correctly, subjects tended to use either a "number facts" strategy or a "plugging-in" (i.e., trying a succession of different values until a suitable one is found) strategy for equations with small coefficients; and for equations with large coefficients either an "undoing twice" (i.e., performing the inverses of the

TABLE 2

Strategies Used in the Solving of Equations

Equation		n	Results	Strategy
6a = 18	①	n = 10	- - -	undoing
			- - - - - - -	number facts
27b = 1053	[illegible]	n = 10	- - - - - - - -	undoing
			-	multiples
			x	undid with wrong operation
n + 5 = 17	⑤	n = 9 *	- - - - - - - -	undoing
			-	number facts
x + 652 = 1348	[illegible]	n = 10	- - - - - - - - - -	undoing
3n + 5 = 11	⑦	n=9	- - -	undoing twice
			- -	number facts
			-	plugging in
			x	used coef. of n for solut.
			-	undoing & no. facts
12x + 216 = 468	[illegible]	n=9	- - -	undoing twice
			- -	plugging in
			- x	undoing & plugging in
			x	undoing w/ same op. & undo.
			x	adding 2 terms on left, then dividing into rt. side
2m - 5 = 7	[illegible]	n=10	- -	plugging in
			-	undoing & no. facts
			-	number facts
			-	undoing twice
			x	reversed (7 - 5 = 2)
			x	changed sign & revers. l.s.
			x	did not try
			x x	undid once, then stopped
17x - 124 = 199	④	n=10	- -	undoing twice
			-	plugging in
			x x x x	undoing w/ same operation
			-	undoing & plugging in
			x	added 124 to 17x2; stopped
			x	did not try
$\frac{x}{3} + 26 = 432$	⑨	n=10	- -	undoing twice
			x x x	undoing only once
			x	common denominator
			x	undoing w/same op. & undo.
			x	undo. once, then div. by 3
			x x	no time

- = correct answer

x = incorrect answer

⊔ = not told that 2c = 2 x c

ⓝ = refers to the order in which the equations were presented

* The tape ran out for one subject

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two given operations) strategy or a combination of both undoing and plugging-in. Though the large coefficients proved to be no hindrance at all for equations of the forms $ax = b$ and $x + a = b$, they tended to increase slightly the complexity of the two-operation equations. A more obvious source of difficulty was the presence of a subtraction sign in an equation. Equations of the form $ax - b = c$ resulted in significantly more errors than equations of the form $ax + b = c$.

The most common error was "undoing with the same operation", for example, subtracting 124 from 199 in the equation $17x - 124 = 199$ rather than adding. This occurred four times in $17x - 124 = 199$, but only once in $12x + 216 = 468$. The reason why this error should occur more often in equations involving subtraction is still not clear. A prevalent error in the last equation, $x/3 + 26 = 432$, was "undoing only once", that is, subtracting 26 from 432, and assigning that difference to $\underline{x}$.

Conclusions. What is s[illegible]ing from this preliminary investigation is that there were so few errors in all, considering the fact that these subjects have not yet been taught any formal algebra. Many of the miscellaneous errors which did occur are being attributed, for the time being, to the confusion caused by the symbolic convention used in these equations. Though strangely, this did not seem to be a factor in $6a = 18$ or $27b = 1053$. Even the six subjects who had not been told that "2c" means "2 x c" seemed to have made this assumption in the $ax = b$ type of equations. However, despite the notational problems, just as children beginning elementary school already possess an informal arithmetic (Ginsburg, 1977), children beginning secondary school seem to know and are able to do some algebra at an intuitive and informal level.

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AN ANALYTICAL FRAMEWORK FOR MATHEMATICAL VARIABLES

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L'emploi de lettres comme représentation symbolique de la notion de variable est utilisé de multiple façons en mathématique. Dépendant du contexte ainsi que des éléments représentés, le rôle d'une variable peut être celui de nom, de valeur positionnelle, d'indice, d'inconnue, de nombre généralisé, d'indéterminée, de variable dépendante ou indépendante, de constante, ou de paramètre. Du point de vue mathématique, le rôle d'une variable est déterminé par le contexte ainsi que les référents et est essentiellement indépendant de la lettre employée pour représenter la variable. Cependant, du point de vue psychologique, l'interprétation d'une variable peut dépendre du choix du symbole, tel que l'ont montré des recherches sur la capacité des étudiants à "conserver" l'equation ou la fonction soumises à des transformations de variable. Un cadre d'analyse incorporant chacun des éléments -- symbole, contexte, référent -- peut être utilisé afin de générer différentes tâches permettant d'étudier la capacité des élèves à identifier les divers rôles d'une variable. Ces tâches, administrées dans des entrevues cliniques semi-standardisées, permettent d'évaluer la clarté et la difficulté associées aux rôles joué par les variables. Les résultats ont une portée importante sur le programme de mathématique, surtout en algèbre.

Literal variable symbols are used in a multitude of ways in mathematics. Depending upon the context in which they occur and the element(s) to which they refer, the role of a variable may be described as that of a name, a placeholder, an index, an unknown, a generalized number, an indeterminate, an independent or dependent variable, a constant, or a parameter. Adding to this complexity is the fact that, generally speaking, different literal symbols can be used to represent the same thing, and the same literal symbol can be used to represent different things. At the same time, certain letters have acquired fixed connotations relative to particular contexts. It is no wonder that students have so much difficulty working with literal variables.

Several recent studies have investigated certain aspects of students' understanding of variables. Küchemann (1978, 1981) has followed up the work of Collis (1975) by identifying six ways that students interpret and use literal symbols. Wagner (1981) has used conservation of equation and function tasks to show that many students harbor two misconceptions about variables: (1) that changing a literal symbol implies changing the referent and (2) that the linear order of the alphabet corresponds to the linear order of numbers. Tonnessen (1980) has investigated college students' understanding of variables; Clement, Lochhead, and Soloway (1979) have studied the difficulties students have in translating verbal statements into symbolic form; and Sachar, Baker, and Miller (1979) have compared students' facility in solving equations with numerical versus literal coefficients.

The purpose of the present paper is to outline a tentative analytical framework for investigating students' understanding of variables. This framework can be used to generate tasks that provide measures of the clarity and difficulty of the different roles of variables. Data from semi-standardized clinical interviews with students may have important implications for the mathematics curriculum, particularly in algebra.

SYMBOL, REFERENT, AND CONTEXT

Like the words of verbal language, the *symbols* for mathematical variables acquire meaning only as they appear in some *context* and represent some *referent* (see Figure 1). As in verbal language, the symbol and its referent determine the *semantic role* of the variable, while the symbol and its context determine the *syntactic role* of the variable. Unlike the words of verbal language, the symbols for mathematical variables are quite freely interchangeable, except in certain contexts, most notably formulas, in which particular combinations of letters have acquired a traditional connotation. Because the symbols for variables are so arbitrarily interchangeable, the context and the referent together, apart from any particular symbol, determine an aspect of variables that is uniquely mathematical. That is, the context and referent determine the *mathematical role* of the variable. All three components -- symbol, referent, and context -- as well as all three aspects -- the semantic role, the syntactic role, and the mathematical role -- combine to contribute to the student's interpretation of variables.

Figure 1 indicates some of the range of variation that occurs in the symbol,

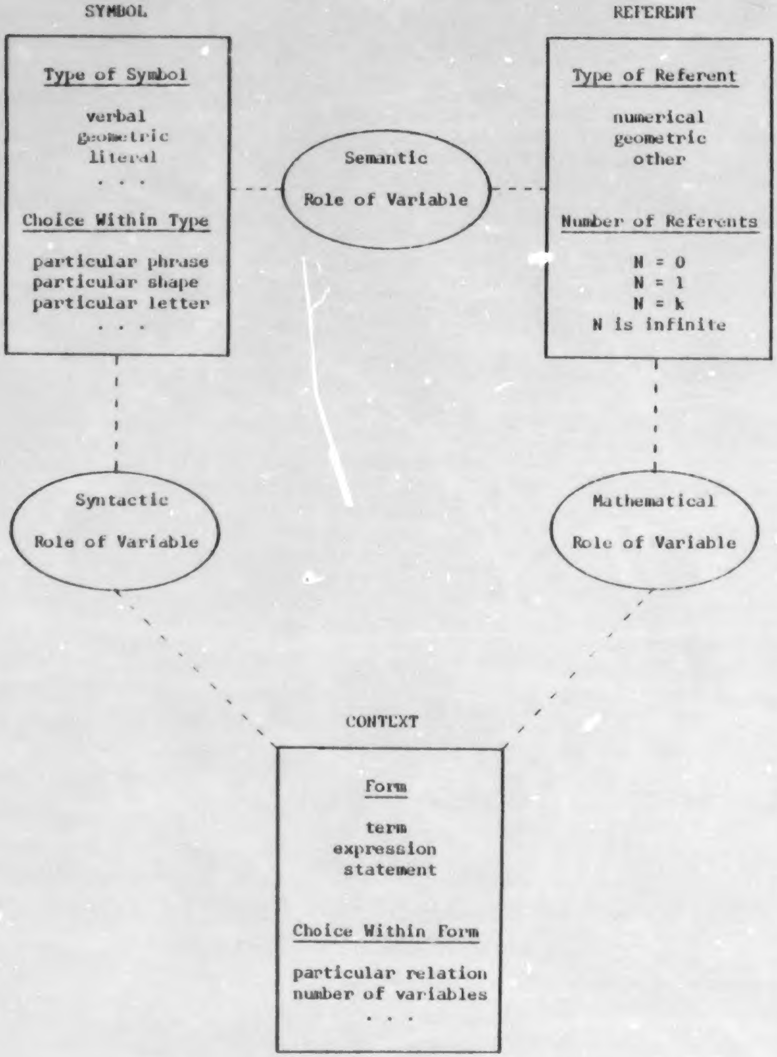
SYMBOL
Type of Symbol
verbal
geometric
literal
. . .
Choice Within Type
particular phrase
particular shape
particular letter
. . .
Semantic
Role of Variable
REFERENT
Type of Referent
numerical
geometric
other
Number of Referents
N = 0
N = 1
N = k
N is infinite
Syntactic
Role of Variable
Mathematical
Role of Variable
CONTEXT
Form
term
expression
statement
Choice Within Form
particular relation
number of variables
. . .

Figure 1. Variation in the components that affect students' interpretations of the roles of variables.

referent, and context components of variables. A change in any one of these components may or may not, depending upon the nature of the change, cause a corresponding change in each related aspect of the variable. That is, a change in either the context or referent may or may not affect the mathematical role of the variable. On the other hand, a change in symbol generally does not affect the mathematical role of the variable, except where conventional usage intervenes. The next three paragraphs provide examples of variation in symbol, referent, and context and show how these variations may affect the mathematical role of variables.

Symbol. Verbal words and phrases, such as "some number" or "an even integer," and geometric figures, such as □ or △ can each be used as symbols for variables in certain roles, but only literal symbols can satisfactorily represent variables across their entire range of roles (Wagner, 1977, 1979). Most choices of literal symbol are quite arbitrary and neither determine nor depend upon the role of the variable. For example, any letter at all can be used to represent the unknown in an equation, a generalized number in an identity, or an indeterminate in a polynomial. In the case of functions, different conventions have prevailed at different times throughout history to facilitate identification of constants and variables. One modern convention is to use letters at the beginning of the alphabet as constants and letters at the end of the alphabet as variables, but even within the constraints of convention, many choices of symbol are available. For example, the sentences $y = ax^2$, $z = bw^2$, and $r = ks^2$ can all be used to describe the same parabolic relationship between two variable quantities. On the other hand, there are some changes in literal symbol that can affect the psychological interpretation of the role of a variable. For instance, changing $y = ax^2$ to $E = mc^2$ will, in most readers' minds, change the constant to an independent variable, the independent variable to a constant, and the parabolic relationship to a linear one.

Referent. A variable symbol may represent virtually any object, person, place, or idea. It may represent one thing, many things, or even nothing at all. Mathematically, it is useful in the case of open sentences to distinguish between the replacement set and the truth set for a variable. Psychologically, the referent set is undoubtedly the truth set. That is, in the equation $3x + 5 = 11$, the referent for the unknown would be the single element of the truth set, the number 2, whereas the replacement set for the unknown would typically be at least the entire set of rational numbers. A change in the specific value(s) of the referent generally does not change the role of a variable, but a change in the number of referents often does change the role of a variable. For example, in the context of equations, x is an unknown in the sentence $x + 2 = 2 + 3x$, but x is a generalized number in the sentence $x + 2 = 2 + x$. On the other hand, x would probably be called an unknown in each of these statements: $x^2 + 6 = 5$, $x^2 + 5 = 5$, $x^2 + 4 = 5$, and $x^2 + 1 = 5$, where the replacement set for x is the set of real numbers.

Context. Like ordinary words in verbal contexts, mathematical variables can occur by themselves, in phrases (terms and expressions), or in sentences (open sentences or statements). Within each of these forms, there are many finer gradations in context. At the level of open sentences, for instance, inequalities differ in context from equations; at the level of equations, a quadratic equation differs from a linear equation; at the level of linear equations, an equation in one variable differs from an equation in two variables. Depending upon the situation, it may be suitable to consider only certain levels of change in context. For many purposes, the finest gradations in context that may be of interest are those involving a change in relation, a change in degree, or a change in the number of variables. Changes in the specific values of numerical terms or factors would not generally be considered a change in context. That is, $x + 2 = 5$ and $x + 3 = 7$ would usually be considered the same context, whereas $x + 2 = 5$, $x + 2 < 5$, $x^2 + 2 = 5$, and $x + y = 5$ would usually be considered different contexts. A change in context may or may not change the role of a variable. For example, the sentences $x + 2 = 5$ and $x^2 + 2 = 5$ may represent different contexts, but x is an unknown in each. Many of the changes in context that do change the role of a variable are accompanied by changes in the referent for the variable, as in $3x + 2 = 5$ versus $3x + 2 < 5$. However, a change in context can change the role of a variable without changing the referent, as shown by the expression $a(b + c)$ and the identity $a(b + c) = ab + ac$; here the variables in the expression would probably be called indeterminates, whereas the variables in the identity would probably be called generalized numbers.

IMPLICATIONS FOR RESEARCH AND TEACHING

The above framework can be used to generate a wide assortment of mathematical expressions and statements that vary systematically in the symbol, referent, and context for their variables. These items can be used to create sorting tasks and triplet comparison tasks to investigate students' ability to identify the different roles of variables. Results could have important implications for the mathematics curriculum. For example, if the notions of placeholder, unknown, and generalized number should be found to be of relatively increasing difficulty for students, then perhaps unknowns should be introduced using the innovative approach developed by Herscovics and Kieran (1980) rather than the more traditional approach of beginning with the idea of generalized number (e.g., Dolciani, Wooton, & Beckenbach, 1980).

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PSYCHOLOGICAL CHANGES ATTENDING A TRANSITION FROM ARITHMETICAL TO ALGEBRAIC THOUGHT

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L'auteur met en discussion les dispositions psychologiques associées vraisemblablement à la manière de penser arithmétiquement et algébriquement. Les réponses de 144 élèves de lycée - dont 24 en chaque année scolaire I-V et 24 élèves préparant l'examen Advanced Level (Baccalauréat) - à deux épreuves mathématiques sont analysées et comparées. La première épreuve demande qu'on produise une solution mathématiquement générale à un problème originalement posé et résolu par Diophante. Trois différents types de solution générale sont présentés et disposés selon une hiérarchie de 'sophistication' (c'est-à-dire subtilité) algébrique. La deuxième épreuve fournit des informations sur la perception de figures géométriques dans lesquelles les distances sont représentées par des symboles algébriques. Les réponses à celle-ci sont divisées en deux catégories - dites concrètes et abstraites - et l'auteur les examine en établissant les correspondances avec les réponses relevées pendant la première épreuve. Cette comparaison soutient la thèse que le niveau de 'sophistication' de la solution présentée par un élève a rapport, étant donné le système symbolique préféré par l'élève, à la manière de percevoir des figures géométriques. Il est probable que la pensée arithmétique ait rapport à une interprétation 'statique' des figures géométriques, tandis que la pensée algébrique se rapporte à une interprétation 'dynamique'.

INTRODUCTION

This paper concerns itself with an initial attempt to distinguish the psychological traits which separate arithmetical and algebraic thought. A theoretical stance was originally taken for task construction which involved an arbitrary but informed decision to consider algebra to begin when the letter is first used as a 'given' to represent a known quantity. Historically this takes place with the introduction of the language of symbolic formalism by Vieta (circa 1600 AD). Reflection upon this innovation in relation to classroom activity suggests the need for a radical change of habitual modes of interpretation developed through prior experience with the language of 'arithmetic with letter appendages', in which the letter is used exclusively as a classical 'unknown'. This language coincides with the period of sycopated algebra which begins with Diophantus (circa 250 AD).

To illustrate key points the responses of 144 eleven to eighteen year old grammar school pupils (24 in each Year 1-5 chosen to represent the range of abilities across each Year-group, and 24 'A' level mathematicians) to two algebraic tasks are presented and analysed. The first task illustrats the different ways in which letters can be used to present a general result, and the second indicates how the pupil interprets geometrical data when distances are given in algebraic terms (a cm, ...).

The first task, the 'Zetetic Task' (ZT), is a reformulation of a problem first posed and solved by Diophantus, and later by Vieta using his more sophisticated language system:

> 'If you are given the sum and the difference of any two numbers
> show that you can always find out what the numbers are. Make
> your answer as general as you can.'

It was presented to each pupil in an interview situation.

The second task, the 'Parallel Lines Task' (PLT), developed from the notion that the same sentential form, word series or term (surface-structure) often serves to convey more than one (deep-structure) meaning (Harper, 1978; Skemp, 1979). It asks the following (intentionally ambiguous) questions about Figure 1:

1. Is the red line longer than the green line, the green line longer than the red line, are they equal in length, or could any of these be possible?
2. Why?
3. When is the green line longer than the red line?
4. When is the red line longer than the green line?
5. When are the lines equal in length?

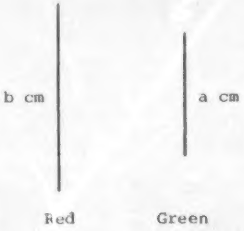

Fig. 1

The task was presented during the same interview situation, and pupil responses were recorded and transcribed.

The first section below presents, compares and cross-classifies solution-types to the two tasks, and the final section uses the data to illustrate a discussion of the likely differences between an arithmetical and an algebraic disposition of mind. Particular areas of attention for future research are indicated.

SOLUTION TYPES TO TWO TASKS

TASK 1 (ZT)   38 Pre-'A' level, and all 24 'A' level pupils produced one of three types of 'general' solution:

(i)   Rhetorical: The pupil typically writes down little except perhaps two numbers to represent a sum and a difference, and the 'solution':

> 'You add the sum and the difference together and divide by two. That gives you one number. Take the difference from the sum and divide by two and that gives you the other number.'

(ii)   Diophantine: The pupil chooses a particular numerical sum and difference, writes down two equations containing two unknowns, and solves them. He (she) then often suggests, verbally or in writing, that the same method can be used whatever the numbers chosen for the sum and difference.

(iii)   Vietan: The pupil writes down two simultaneous equations involving two unknowns and a letter for each of the sum and the difference. These are solved to produce, for example: $x = \frac{a + b}{2}$, $y = \frac{a - b}{2}$.

Table 1 shows the distribution of solution-types.

	Number of solutions in each Year group					
Solution	Year 1	Year 2	Year 3	Year 4	Year 5	'A' level
type	11:9	12:11	13:9	15:0	15:9	17:3
R	4	4	4	1	2	0
D	0	1	3	5	5	4
V	0	1	0	1	7	20

Table 1: Distribution of solution-types to the ZT

Observations

(a) Some younger pupils are algebraically more sophisticated than their older peers.

(b) The figures indicate an age-related transition Rhetorical → Diophantine → Vietan.

TASK 2 (PLT)   Responses were divided into two classes:

(i)   Concrete: Pupils suggest the red line is longer because 'It looks longer', 'It measures more', etc. Arithmetico-concrete operations are suggested in response to the remaining questions. For example, 'Double the green line', 'Chop some off the red line', 'Bring the green line nearer'.

(ii) Abstract: Relationships between letters are introduced: 'When a > b', 'When b > a', 'When a = b'.

Table 2 shows the distribution of response-types.

Response type	Number of responses in each Year group					
	Year 1	Year 2	Year 3	Year 4	Year 5	'A' level
	11:9	12:11	13:9	15:0	15:9	17:3
Concrete	22	19	18	18	16	4
Abstract	2	5	6	6	8	20

Table 2: Distribution of response-types to the PLT

Observations

(a) The geometrical data is perceived in two, and perhaps three, distinct ways: (i) the lines are singular objects with measurable lengths; (ii) the lines have 'unknown' lengths yet to be found; (iii) the lines are attempts to represent 'givens' and so do not have a length in the measurable sense. (Future research needs to discover a way of deciding which of (ii) and (iii) is intended by the 'Abstract' response).

(b) The algebraic data is perceived in two, and perhaps three, distinct ways: (i) the letters are given content and ordering by the geometrical data; (ii) the content and ordering of the letters remains 'unknown' despite the geometrical data; (iii) the letters are conceived of as 'givens' from the outset and so a (iii) above applies.

Cross-classification of results

A cross-classification of the results of Tasks 1 and 2 produces Table 3 (upper triangle: Years 1-5; lower triangle: 'A' level figures added):

		ZT			
		Non-general	Rhetorical	Diophantine	Vietan
PLT	Concrete	75 75	11 11	6 7	1 4
	Abstract	7 7	4 4	8 11	8 25

Table 3: Cross-classification of PLT and ZT responses

Observations

(a) The greater is the sophistication of the ZT solution the more likely is an

'Abstract' response to the PLT.

(b) An 'Abstract' response to the PLT does not preclude a non-Vietan, nor a non-general response to the ZT and vice-versa.

DISCUSSION

Assuming arithmetic to comprise 'numerical calculations' there are changes along at least two dimensions to take into account with respect to the development of algebraic thought:
(i) the presence or absence of an 'analytic' ability (Krutetskii, 1976) which here underpins the production of general solutions to the ZT; and (ii) the variety of usages made of the letter (Collis, 1975; Kuchemann, 1978).

Each type of general solution demonstrates the presence of the 'analytic' ability, in the sense that significant aspects of a problem are isolated, 'knowns' and 'unknowns' identified and related, and at some stage what is known perceived as an exemplar of the general.

Table 1 indicates that a symbolism is eventually accepted to support this ability. In this small scale survey the most common early usage of the letter is as an 'unknown' in the ZT. Later the letter is also used as a 'given' to represent known quantities, although the extent to which the sequence 'unknown' followed by 'given' can be changed by teaching is yet to be resolved.

It is thus only at the final 'Vietan' stage that the ability to perceive the general in the particular is formalised in symbolic terms through the introduction of the letter as a 'given'. In this sense cognition and symbolism are, until this time, out of phase one with the other.

Table 3 indicates that the incidence of 'Abstract' responses to the PLT is related to the symbolism available. There are sound theoretical reasons for this correspondence - in particular the fact that the pupil who responds to the letter by interpreting it as an 'unknown' will be psychologically predisposed towards 'finding' values, and thus more likely to be influenced by attendant geometrical data than will the pupil who uses the letter especially as a 'given'. What the Table shows is that when the stage is reached at which the 'given' is used spontaneously in the ZT (which almost certainly involves a different form of ALC to that identified by Collis (1975a)) pupils almost invariably ignore geometrical orderings in the PLT. In turn it is thus possible that through the synthesis of the 'analytic' ability and the relevant symbolism the pupil 'perceives the general in the particular' also in respect of geometrical data, i.e. either perceives a drawing such as Figure 2 as an exemplar of an infinite class of possibilities, or as a 'snapshot' of a dynamic system (Harper, 1980).

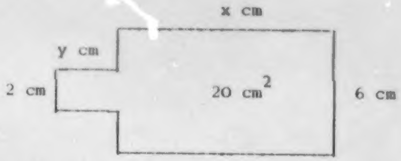

Fig. 2

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THE EFFECT OF VARYING THE PERCENTAGE OF CLASS TIME SPENT ON DEVELOPMENTAL AND PRACTICE ACTIVITIES IN FIRST YEAR ALGEBRA

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Abstrait

L'effect de la variation du temps passé aux activitiés du dévéloppement et du pratique des rudiments d'algèbre

L'effet de la variation du temps passé aux activitiés du developpement et du practique des rudiments d'algèbre a été examinés (investigués) avec quinze classes d'etudiants au troisième dégrée dans cinq écoles. L'analyse des résultats des moyennes ajustées des classes sur un instrument de trois parties (la connaissance, les habilités et la solution des problèmes oraux) a été faite sur les examens de retention immédiat et diffère.

Des differences significatives ont été trouvés en faveur du groupe qui a dévoué la plupart du temps en classe en faisant des activitiés basé selon le sous-examen de la retention de la solution des problems oraux.

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INTRODUCTION

This study examined the effects on students' achievement and retention of varying the percentage of class time spent on developmental activities and on practice activities in first year algebra classes studying topics such as factoring polynomials, multiplying polynomials, and related topics. By developmental activities in this study we shall mean those activities which are designed to induce meaning and understanding as specified by the objectives. The role of the teacher in this type of teaching is to select instructional strategies and arrange for activities which involve the students in an active fashion whereby conclusions are made by means of thinking processes. By practice activities in this study we shall mean those activities which are designed to maintain and improve basic skills and understanding of concepts with emphasis being placed upon increasing proficiency and recall. All these activities assume that the learner has an understanding of the material and the purpose of the practice is to embed or permanently fix the skill or idea in the student's cognitive structure.

The significance of this study was in its production of evidence upon which decisions concerning time utilization might be made in such a way as to determine a most effective balance between developmental and practice activities in mathematics instruction. In order to produce a maximum of both understanding and mastering of basic skills, such decisions should be based on experimentation. If a large amount of time is devoted to either category of activities, then, only a small amount of time is available for the other category. Consequences of imbalance in time utilization for these two categories are easily formulated. For example, a student might know a method (be able to carry out an algorithm) and yet neither be able to identify the nature of the problem, nor be able to translate the problem into a mathematical model, thus finding it impossible to apply the known method. This situation may be referred to as problem solution ability without comprehension of mathematical principles ability when a student will say "I could solve the problem, if only I could set it up." On the other hand, if the instructor devotes too much time to meaning the development of skills suffers and the student may also have problems. For example, a student may be able to identify the nature of a problem, translate the problem details into a mathematical model, but not be able to finish the problem because of inability, i.e., lack of skills, to carry out the necessary algorithms. The data from this study is significant in that it provides some empirical data bearing upon the delicate balance between practice and development. Moreover, this study is also significant because it aids in the void relative to similar research at the secondary school level. Previous research by Shipp and Deer (1960), Shuster and Pigge (1965), Hopkins (1965) and Zahn (1966) focused on the elementary school level and results tended to show that more time should be spent on developmental activities than on practice activities. Though Zahn's subjects were eighth graders, who in many cases take first-year algebra, the results tended to indicate that studies of this nature should be considered at the secondary level. And as first-year algebra tends to be the course in which practice and/or developmental activities appear to be crucial for more advance work in mathematics that course seemed to be a logical place to begin.

SUBJECTS AND TREATMENTS

The study was a controlled experiment involving three experimental groups of ninth grade students (N = 324) enrolled in 15 sections of first-year algebra. There were about the same number of girls as boys enrolled in each of the 15 sections of first-year algebra. Students enrolling in any section of first-year algebra were required to have a grade point average of at least B or favorable teacher's recommendations.

The 70/30 group consisted of five distinct sections of first-year algebra students who were taught by five distinct teachers in a fashion where 70% of the instructional time within each 50 minute class period was devoted to developmental activities and the remaining 30% was devoted to practice activities. The 30/70 group consisted of five distinct sections of first-year algebra students who were taught by the same five teachers who taught the 70/30 group in a fashion where 30% of the instructional time within each 50 minute class period was devoted to developmental activities and the remaining 70% was devoted to practice activities. The control group consisted of five distinct sections of first-year algebra students who were taught by five distinct teachers in a fashion that did not consider a specified amount of instructional time to developmental and practice activities.

DATA COLLECTING INSTRUMENTS AND RESEARCH PROCEDURE

The tests used to measure achievement and retention in this study were composed of subtests measuring knowledge (definitions, properties, symbols, formulas and specific facts); basic skills (performing algebraic operations, demonstrating the meaning of terms, and in general "doing" types of things); and problem solving (verbal problems typically found in first-year algebra textbooks). The tests were researcher constructed in two parallel forms by subtests; Kuder-Richardson-20 reliability coefficient estimates for the subtests ranged from .74 to .86. Correlations between the scores on two administrations of the test one day apart showed the stability of each of the subtests to be .80 or greater. Moreover, the subtests were considered to be parallel in nature by using the "t-test" in determining that no significant difference existed between the means of the subtests ($p < .05$). Also the variances were shown to be equal by subtests using the F_{max} test ($p < .05$).

Further, to insure face validity of the tests the following National Assessment of Education Progress (1970) philosophy was used in this study:

> An exercise has content validity if it is a direct measure of some important bit of knowledge, skill or attitude that reflects one or

> more objectives of a subject area. It must not be trivial, inconsequential or peripheral to the objective. In particular, then, an exercise has content validity if it makes sense to an informed reader who sees it together with an objective and says, "Yes, this is a good measure of the knowledge or skill called for by this objective." (p. 15)

Using the content outline and the textbook for the first year algebra course, behavioral objectives were written for the content to be taught during the study. A pool of items was then constructed to assess these objectives. For each objective four potential items were placed in the item pool. Content validity was established by having two mathematics educators and one mathematician at a major mid-western university (U.S.A.) review the content outline and judge whether the potential items fitted the stated objective. Further, each potential item was required to receive approval of at least two of the panel members, otherwise it was rejected. After receiving the items from the panel four items were picked for each objective. They were constructed in pairs such that one pair could be answered correctly by 50% or more of the students and the other pair could be answered correctly by less than 50% of the students. A toss of a coin determined which pair would be used to assess the given objective and another toss determined which item would go on a particular form. Choosing items in this fashion for the two forms resulted in tests that measured achievement quite well over a fairly wide range or levels. Twenty-five items were placed on each form. Items 1, 3, 6, 8, 11, 13, 16, 18, 21, and 23 were knowledge items; items 2, 4, 7, 9, 12, 14, 17, 19, 22, and 24 were skill items; and items 5, 10, 15, 20, and 25 were problem solving items. The knowledge items were used to compose the knowledge subtests, the skill items were used to compose the skill subtests, and the problem solving items were used to form the problem solving subtests.

The item discrimination index used was the non-spurious point biserial correlation coefficient. This is a measure of correlation between a discrete variable and a continuous variable, in this case, the test item and the test score. The average point biserial across items for the knowledge and skills subtest was .4 and was .5 for the problem solving subtest for both Forms A and B.

To maintain control of this study the 70/30 and 30/70 treatments were randomly assigned and regulated through lesson plans prepared by the investigator, which allowed the five teachers to use their regular textbooks and the course content schedule. The plans were constructed in pairs with the same set of objectives appearing on each plan insuring the same content was covered under both treatments. Further, the teacher maintained a daily log on the amount of time spent on each activity; regular classroom observations were conducted by the investigator to insure the treatments were being followed.

The research procedure followed was to acquaint teachers with the study, pretest the groups of students, instruct using the assignment treatments, then posttest the groups of students to determine the effects of the treatments on achievement. To ascertain retention the students were retested 7 weeks after the posttesting. The instructional period consisted of 32 school days.

Analyses of covariance were used with adjusted class means as the unit of analysis. Total mathematics scores on the Iowa Test of Basic Skills and subtest scores from the various pretests were used as covariates. The Newman-Keuls Multiple Comparison Test was used for post hoc analyses.

FINDING AND CONCLUSION

1) A statistically significant amount of learning took place in each of the three experimental groups (70/30, 30/70 and control) in the three areas measured (knowledge, basic skills and problem solving) p .05. 2) From classroom observation it was determined that the control group devoted 25% of their class time on development and 75% to practice. 3) There was no significant posttest or retention test differences between groups on the knowledge and basic skills subtests. 4) Both the 70/30 group and the 30/70 group scored significantly higher than the control group on the problem solving posttest. On the problem solving retention subtest, the 70/30 group scored significantly higher than the 30/70 group and the 30/70 group scored significantly higher than the control group ($p < .05$).

On the basis of the current research and previous research on instructional time, it appears that increased attention should be given to developmental work at the secondary level. More research is needed before precise conclusions can be drawn. For example, it may be that the level of intellectual development of students is a factor which influences where cognitive gains (knowledge, skills, problem solving) may accrue from increased time on developmental work.

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FUNCTION CONCEPTS: INTUITIVE BASELINE *

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Selon Fischbein, des intuitions sont des représentations mentales de faits évidents. En enseignant, le maître doit être conscient des intuitions présentes chez ses élèves au début du procès d'apprentissage. Ce travail présente une étude, dont le but était d'établir quelle est la base intuitive chez des élèves de la sixième à la neuvième année scolaire, pour un nombre de concepts reliés à la notion mathématique d'une fonction.

Afin de tenir compte de la complexité de la notion de fonction, le modèle suivant a été développé: Les concepts associés avec la notion de fonction (image, zéro, croissance, etc.) apparaissent souvent dans des représentations particulières (graph, table de valeurs, etc.) et à de différents niveaux d'abstraction. En arrangeant ces représentations, concepts et niveaux dans une structure trois-dimensionelle, on obtient un cadre théorique propice à la formulation systématique de questions et hypothèses sur les bases de l'apprentissage, son cadre, son transfer, etc.

Un projet de recherche a été élaboré en suivant ce cadre théorique. Est relatée ici, la première phase, dans laquelle les intuitions d'élèves sur les concepts d'image, de préimage, de croissance et de points extrémaux ont été examinés dans trois représentations (graphs, tables de valeurs et diagrammes de flèches) et à deux niveaux d'abstraction (concrèt et abstraît). A ce but, trois versions d'un même questionnaire ont été construites, chacune contenant les mêmes questions dans une des trois représentations. Les questionnaires ont été distribués

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* This research has been administered through the Department of Science Teaching, The Weizmann Institute of Science, Rehovot, Israel.

parmi 443 élèves d'aptitude mathématique variée et provenant de différents niveaux sociaux. Les principaux résultats étaient

(i) les intuitions sont indépendantes de la représentation.

(ii) les intuitions progressent avec l'âge (sauf de la $7^{\text{ème}}$ à la $8^{\text{ème}}$ année scolaire); ce progrès est plus fort chez les filles que chez les garcons.

(iii) Les différences entre les niveaux (sociaux et aptitudes) sont beaucoup plus prononcées chez les garçons que chez les filles.

(iv) Les résultats pour les deux niveaux d'abstraction sont semblables.

L'extension a d'autres concepts fonctionels et l'examination du transfer entre différentes représentations sont prévues pour des phases ultérieures du projet de recherche.

INTUITIONS

For the purpose of this study, Fischbein (1973) will be followed in so far as the term "intuitions" is taken to refer to mental representations of facts that appear self-evident. For example, to most junior high school students, it appears self-evident that:

i) the whole is equal to the sum of its parts and greater than any one of them.

ii) for real numbers a, b, c: if $a > b$ and $b > c$, then $a > c$.

On the other hand, the following statements do not appear as self-evident, even to most college students:

iv) there are as many points in a line of length ℓ as there are in a line of length 2ℓ.

v) the graphs of two quadratic equations can intersect in more than four points.

THE FUNCTION CONCEPT

The difficulties associated with the teaching and learning of the function concept have been studied by Thomas (1975) and Dorfeev (1978). The concept is a complex one. Several reasons for this are:

1. The function concept is not a single concept by itself, but has a considerable number of subconcepts associated with it (e.g. domain, image).
2. The function concept is being used to tie together seemingly unrelated subjects. In going through the associated abstraction process, different levels of abstraction are encountered (e.g. number of variables, type of domain and range).
3. The same function may be represented in a number of different settings (e.g. a table, arrow diagram, graph, formula, verbal description).

Because of the intrinsic structure associated with the nature of function, the subject "functions" can be thought of as being arranged in a three-dimensional block type structure, in which the x-dimension carries the various settings, the y-dimension the function concepts and the z-dimension carries a taxonomic scale of levels of abstraction and generalization.
Within this framework, horizontal transfer of learning (transfer of a concept learned in one setting to another setting) now appears as movement parallel to the x-axis of the function block, whereas vertical transfer of learning (transfer to levels of greater generality) appears as movement parallel to the z-axis. Progress parallel to the y-axis of the block corresponds to the learning of new concepts and therefore cannot in general be expected to occur without an external stimulus.

The function block thus provides a framework within which to systematically ask questions concerning the ordering, arrangement and presentation of function curricula.

Fishbein (1973, 1979) has shown the necessity of developping curricula which build upon intuitive support. This study examined the intuitive support which can be exploited by the teacher, when approaching the concept of function.

EXPERIMENTAL DESIGN

Three questionnaire booklets were constructed in which questions were asked on the concepts of image, preimage, growth, extrema and slope. The three booklets contained the same functional relationships and were identical except for the settings in which the functional relationships were presented: either a graph or a diagram or a table setting. Two functions were presented in each booklet; one concrete, giving a pedestrian meaning to the relationship; the other abstract, removed this pedestrian meaning.

The booklets consisted of 42 multiple choice questions. All questions included in the booklets passed an external validity test. The questionnaire had an internal reliability (KR20) of 0.91.

The booklets were given to 44[illegible] pupils in grades 6 - 9 who were classified as being of a high or low level. (The level is a construct variable taking into account ability and social factors. This variable will henceforth be called ABSOC.)

FINDINGS

One sees in Figure 1, that the intuitions of pupils progress considerably from grade 6 to grade 9 with a stagnation from grade 7 to grade 8.

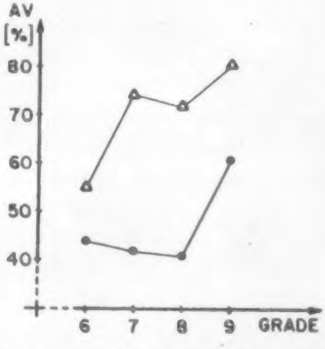

Fig. 1: **AVERAGE BY GRADE FOR HIGH (△) AND LOW (•) ABSOC**

Moreover, 9th graders with low ABSOC were essentially two and a half years behind those with high ABSOC. This is important to note. Lewy & Chen (1977) claim that with respect to general cognitive performance, socially disadvantaged pupils can learn the material, it simply takes them longer to do so. Although the variable ABSOC is not identical with the percentage of socially disadvantaged, Fig. 1 appears to support the conjecture that Lewy and Chen's observation holds not only for general cognitive performance but also for intuitions.

The overall mean scores of the males vs. females were, for all practical purposes, equal (58% (male) vs. 59%). However, in grades 6 and 7 the boys exhibited more intuitions on the functional concepts whereas in grades 8 and 9 this was reversed and the girls did better. Recalling the lack of progress in performance from grade 7 to grade 8, one sees that quite a bit is happening under the surface. It turns out, that a similar "switching" occurs in the interaction between ABSOC and Sex (see Fig. 2). The boys perform more extremely than the girls, low level boys performing worse than low level girls and high level boys performing better than high level girls.

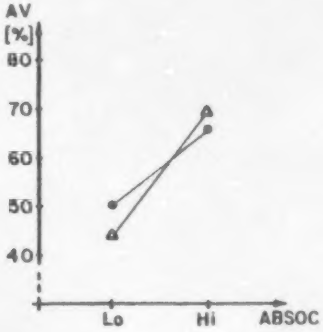

Fig. 2: **AVERAGE BY ABSOC FOR FEMALE (•) AND MALE (△) SEX**

The above results carried over to both the abstract and concrete sections of the test. Comparing the mean scores achieved for the various functional concepts in the questionnaire, it was observed that questions on the concept of image were answered best and questions on the concept of slope were answered worst. It is, however, more interesting to compare differences between the three versions of the booklet for each concept. Pupils with High ABSOC preferred the graphical setting throughout for all concepts, whereas low ABSOC pupils preferred the table setting. Didactically this suggests that the subconcepts should be introduced in a graphical setting for high level students and in a table setting for low level students.

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DIFFICULTIES RELATED TO THE CONCEPT OF VARIABLE PRESENTED GRAPHICALLY

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Lorsqu'elle est présentée à l'aide d'un graphique cartésien, la notion de variable entraîne des difficultés qui sont tantôt d'ordre sémantique tantôt d'ordre symbolique (ou des deux). Nous allons présenter les données recueillies auprès de plus de 300 élèves du niveau secondaire. Des interviews préalables ont permis une catégorisation poussée des réponses. Deux questions seront surtout étudiées: les problèmes reliés à l'idée de croissance et le comportement de discrétisation.

L'idée de croissance rapide est confondue tantôt avec "être grand" (attraction vers les grandes valeurs), tantôt avec "commencer à croître" (attraction vers le bas). Ces attractions peuvent être faibles ou fortes. En des circonstances diverses, les élèves montrent une tendance à percevoir le continu comme un ensemble discret d'éléments. On examinera, entre autres, la cohérence des réponses (corrélation) et leur évolution à travers les niveaux.

Note: La présentation sera faite en anglais avec documentation disponible en français car le sujet aura été traité plus en détail au séminaire de didactique des mathématiques à Paris en mai 1981.

INTRODUCTION

When presented graphically the concept of variable gives rise to various difficulties which are either at the semantic or the symbolic level (or both). We shall present data obtained from over 300 pupils, more than 100 from three age groups: 12, 14 and 16 years old. Previous interviews have allowed a rich and complex classification of responses. A computer compilation will be used for data analysis and correlations.

A GENERAL PRESENTATION OF THE TASKS

The increase of a variable (dependent) over an internal is represented graphically by a cartesian graph having roughly the shape shown in figure 1. The pupils were asked to find when the increase is the greatest (in the terms related to the situation) in three different contexts. They were also asked in one context when the variable starts to increase.

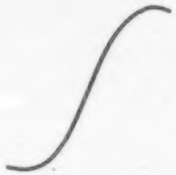

Figure 1

Two variables can be represented on the same graph by two intersecting curves as the ones shown in figure 2. Pupils were then asked when one variable is bigger than the other (with situational expressions).

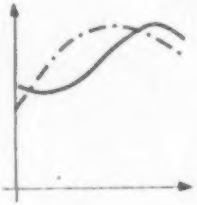

Figure 2

RESPONSES

A) GROWTH

Let us characterise four important responses (incorrect):

- Attraction by high values.

  We have decided to give such a name to a recurrent mistake which consists of answering by a value or an interval corresponding to the upper part of the curve.

  The attraction can be strong. The pupils then confuse "be big" and grow rapidly. We observed such a difficulty with problems in which are involved more than graphical distractors.

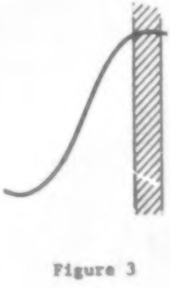

Figure 3

The attraction can be weak. Pupils then answer with intervals which tend to include high values or with values near the maximum. In that case interviews suggest that even though pupils may "look for the steepest" the shape of the graph plays a deceiving role on the perceptual evaluation of the gradient (deception certainly due also to the insidious and conflictual schema of "being high").

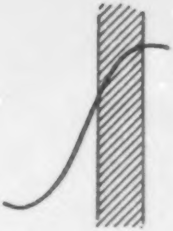

Figure 4

- Attraction by low values.

  This attraction is characterised by values or interval located at the lower part of the curve curve. The attraction can be strong. The pupils then confuse "start to rise" with "rising most rapidly" and give value very near the minimum.

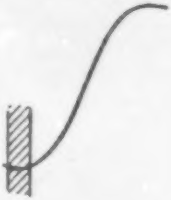

Figure 5

The attraction can be weak. The mistake seems to be more subtle. We incline to believe that pupils refer to "start to grow quickly" or "start to grow for real". The shape is certainly, with what it suggests, determinant.

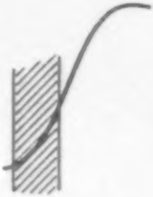

Figure 6

B) DISCRETISATION

This term refers to many mistakes by which pupils show an inclination to break down lines or curves into discrete parts (not dense or continuous) or to jump or switch to any kind of "whole values".

- Description of an interval.

  When determining "when" a variable is greater than another one, they describe the interval point-wise using one or more points inside the interval.

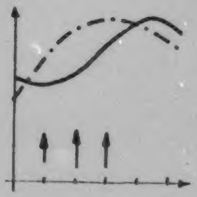

Figure 7

- Switching to whole numbers.

  In reading processes, pupils in various ways often use only the graduation numbers as if nothing else exists on the axis or on the curve.

- When does it start rising?

  An interesting case we discovered is related to this question. In fact, many pupils refuse to answer the value corresponding to the minimum (which is sensible) but refer to the next whole value.

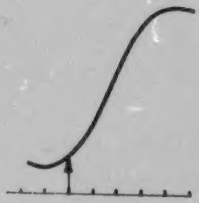

Figure 8

CONCLUSIONS

At the conference, we shall present the graphs which we used in the questionnaire and examine the data obtained in order to

- examine the evolution of those responses with the ages;
- establish the coherence of those responses by examining correlations between answers provided in various contexts;
- establish tentative ranking and evolution of the responses by examining the patterns of displacement of the pupils between the categories*: for example, the category "attraction by low values" appears to be very stable.
- put forward a few hypotheses on difficulties inherent the concept of variable presented by means other graphical.

* In fact, we have for the pupils aged 12, pretest and post-test results providing us with interesting cross-tabulations.

THE QUADRATIC FUNCTION AS A VEHICLE FOR DISCOVERY BY DEDUCTION

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Skemp attire l'attention sur la confusion entre l'approche logique et l'approche psychologique de l'étude des mathématiques. A son avis, la première approche enseigne la pensée mathématique, tandis que la seconde met l'accent sur le raisonnement mathématique. Il déclare aussi qu'un des désavantages de l'approche logique est qu'elle ne donne que le produit fini de la découverte mathématique, et ne réussit pas à faire faire à l'élève les processus de la pensée, par lesquels se font les découvertes mathématiques.

En général, ce point de vue est probablement juste, mais en pratique, il est peu probable que la séparation soit si claire, et nous désirons montrer par un exemple que des aspects de l'approche logique et de l'approche psychologique peuvent s'accorder avec succès, et ne sont pas nécessairement en contradiction les uns avec les autres. Nous fournissons une infra-structure dans laquelle l'élève fait ses propres découvertes dans le cadre d'une structure déductive qu'il aide à créer.

L'exemple: Un bon élève de 9ème année scolaire a une bonne base en ce qui concerne des concepts tels que les fonctions en général, les graphes et les fonctions linéaires en particulier. Le "chapitre suivant" est la fonction du second degré qui est, en général, présentée à l'élève, ni très logiquement, ni très psychologiquement. Par contraste, nous décrivons une approche des fonctions du second degré qui

1) se fonde naturellement et essentiellement sur les connaissances de base de l'élève;

2) est déductive et a une forte structure logique; et

3) dirige l'élève à "prouver et à découvrir" les résultats et ensuite à les énoncer explicitement.

Cette approche prend en considération la maturité du raisonnement mathématique de l'élève et la développe. Un aspect inhabituel à cet age est que le développement procède du général au particulier; cela veut dire que la structure déductive se rapporte à la fonction du second degré en général et laisse la discussion de fonctions particulières du second degré à l'acquisition des techniques à un stade ultérieur

Skemp draws attention to the confusion between the logical and the psychological approaches to mathematics learning, and suggests that the former teaches mathematical thought, whereas the latter emphasises mathematical thinking. He also states that one of the faults of the logical approach is that "it gives only the end-product of mathematical discovery and fails to bring about in the learner those processes by which mathematical discoveries are made" (Skemp, 1971).

In general, this view is probably a correct one, but, in practice, the division is unlikely to be quite so sharp, and we would like to show by an example, that aspects of the logical and psychological approach can be successfully harmonised and are not necessarily in contradiction. We provide a framework in which the student makes his own discoveries within a deductive structure which he helps to create.

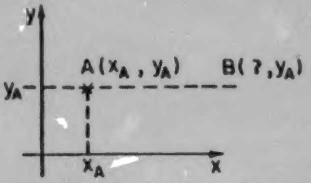
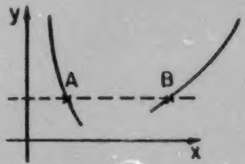
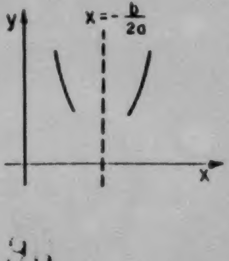
In the example, which is concerned with an approach to the teaching of quadratic functions, we wish to stress three points:

(i) The approach is built naturally and essentially on the students' background which, briefly, consists of a familiarity with linear functions, including the relevant technical skills, the function concept in general and the various representations of functions. The student has met the concept of symmetry and knows how to find the equation of the line of symmetry between two points on a line parallel to the x or y axes.
The next topic, entirely new to the student, is the quadratic function, its graphic representation, and the solution of quadratic equations and inequalities.

(ii) The approach described below was developed for the more able students (the upper third of the population). It has a strong logical, deductive structure which unifies the whole topic.

(iii) The most significant aspect of the approach is that it does not "give only the end-product", as in traditional deductive mathematics (theorem, proof; theorem, proof; and so on ...). The logical deductive structure provides the mathematical skeleton, around which the student adds the flesh by a chain of mathematical discoveries prompted by leading questions.

STAGE I - DISCOVERING THE GRAPH OF THE QUADRATIC FUNCTION

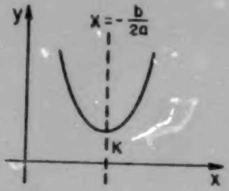
LEADING QUESTIONS	THE MAIN MATHEMATICAL DISCOVERIES AND CONCLUSIONS
1) Does $y = ax^2 + bx + c$ represent a function (from the real numbers to the real numbers)? What can you conclude about its graph?	1) $y = ax^2 + b + c$ represents a function. The graph cannot "turn back" on itself; or, a line parallel the y-axis meets the graph in not more than one point.
2) Given any point A (x_A, y_A) on the graph of $y = ax^2 + bx + c$, are there any other points $B(x_B, y_A)$ on the graph with the same y-coordinate?  What can you conclude about the graph?	2) There is another point B with x-coordinate $-x_A - \frac{b}{a}$. $\Downarrow$ The graph has two branches  $\Downarrow$
3) For any point A (x_A, y_A) on the graph, find the midpoint $M(x_M, y_M)$ of AB. What can you say about M for different AB? What can you conclude about the graph?	3) M has coordinates $(-\frac{b}{2a}, y_A)$ $\Downarrow$ M always lies on $x = -\frac{b}{2a}$ $\Downarrow$ The graph of $y = ax^2 + b + c$ has a symmetry with respect to the line $x = -\frac{b}{2a}$ 

$\Downarrow$

There is just one point K common to the graph and its axis of symmetry.

$\Downarrow$

The branches of the graph must steadily increase or decrease from K (the function is monotonic on either side of the axis of symmetry).



4) Find the coordinate y_K of the point K on the graph for which $x_K = -\frac{b}{2a}$.

5) A (x_A, y_A) and B (x_B, y_B) are symmetrical points on the graph of $y = ax^2 + bx + c$

We can write $x_A = x_K + \alpha \quad (\alpha \neq 0)$

$$x_B = \ldots$$

Find $y_A = y_B$ in terms of y_K, α and the coefficients of the function.

What you can conclude about the graph?

$\Downarrow$

$$y_K = c - \frac{b^2}{4a}$$

5)

$\Downarrow$

$$x_B = x_K - \alpha$$

$\Downarrow$

(after a little algebra)

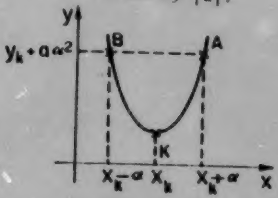
$$y_A = y_B = y_K + a\alpha^2$$

$\Downarrow$

$a > 0$, K is a minimum

$a < 0$, K is maximum

The relative shape of the curve is determined by $|a|$.



STAGE II - FINDING THE ZEROS OF $y = ax^2 + bx + c$

LEADING QUESTIONS

THE MAIN MATHEMATICAL DISCOVERIES AND CONCLUSIONS

6) What conditions must a, b and c satisfy for the graph to intersect, (touch, not intersect) the x-axis?

$\Downarrow$

6) The position of K $\left(-\frac{b}{2a}, c - \frac{b^2}{4a}\right)$ determines the answer:

$\Downarrow$

algebraically

intersect	$4ac - b^2 < 0$
touch	$4ac - b^2 = 0$
not intersect	$4ac - b^2 > 0$

$\Downarrow$

) Find a formula for the zeros of the quadratic equation

$ax^2 + bx + c = 0$

7) If A and B are the zero points,

$$y_A = y_B = y_K + a\alpha^2 = 0$$

$$\Downarrow$$

$$\alpha = \pm\sqrt{\frac{-y_K}{a}} \qquad x_{A,\,B} = x_K \pm \sqrt{\frac{-y_K}{a}}$$

$$\Downarrow$$

$$x_{A,\,B} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

It is important to notice that the algebraic technicalities are much simpler than in the classical methods of developing this topic. No "completing of the square" or "factorisation" of quadratic trinomials is required.

n the above we have described an example of a learning situation which is both bstract and very general. The example illustrates the concept of "deductive iscovery"; that is, discovery within a deductive structure. The student works ith general representatives of classes of mathematical objects; $y = ax^2 + bx + c$ s a representative of the class of quadratic functions, and $A(x_A, y_A)$ as a epresentative point of the class of points on the graph of the quadratic function e is led to discover the properties of these mathematical objects and is expected o become aware of the fact that these properties belong to all "elements" in the lass represented. Only later does he go through the process of concretization -

"the reverse process of abstraction" (Dienes, 1963) - from the class to concrete elements, in order to achieve mastery in the relevant mathematical skills. The first enthusiastic teachers who wished to adopt this approach were nevertheless afraid to teach it at this level of generalization, and fought with us to try it the opposite way round - i.e. by "inductive discovery" - from concrete to abstract representative objects. As a compromise, we developed an approach which was still deductive, but in which the student first works with a concrete example of a quadratic function. We retained the use of a general representative point A (x_A, y_A) on its graph. Subsequently, he develops the full deductive structure using the general representative quadratic function. In the class trials, both students and teachers expressed intellectual satisfaction. Observation further convinced us that the original, completely general and abstract "deductive discovery" approach had considerable chance of success.

Learning by discovery has been discussed and researched a great deal (Ausubel et al, 1978; Bruner, 1964; Egan and Greeno, 1973; Karplus, 1973, and many others). Most of what has been published relates to "inductive discovery". The example above relates to "deductive discovery", which would seem to us to have a considerably different rationale. It is obvious that this sort of "guided deductive discovery" is suitable only for those individuals that have the ability and maturity to deal with the requisite logical and abstract arguments. And in this perhaps, lies the reason for the lack of literature and research - we are dealing with a learning situation which is more complex than that to which the "inductive discovery" method is usually applied - more complex both in its psychological as well as its mathematical nature.

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STRATEGIES AND ERRORS IN SECONDARY MATHEMATICS - THE ADDITION STRATEGY IN RATIO

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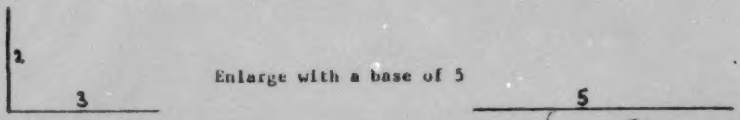
ABSTRACT: Les Stratégies et Erreurs en Mathématiques Secondaires - Proportion

Les recherches des Concepts in Secondary Mathematics and Science (à Chelsea College) au sujet de la proportion ont révélé qu'un tiers des élèves des classes secondaires qu'on a examinés (n=3,000) a donné a quatre des questions les plus difficiles des réponses qui correspondaient à l'emploi de la "stratégie d'addition" identifiée par Piaget (1967) et par Karplus (1975). Le nouveau projet de recherche financé par le Social Science Research Council, qu'on a appelé Strategies and Errors in Secondary Mathematics, a choisi comme un des sujets d'étude l'utilisation de cette fausse stratégie par laquelle l'enfant emploie une différence 'a-b' plutôt que la proportion a/b, et additionne la différence pour obtenir un agrandissement. On a interrogé quarante enfants âgés de 12 à 16 ans qui avaient donné à l'écrit des réponses qui paraissaient démontrer l'emploi de la "stratégie d'addition" afin de découvrir les procédés qu'ils ont employés quand ils one réussi à résoudre un problème de proportion et si au fait ils ont calculé 'a-b' et non a/b dans les questions les plus difficiles. Quelques individus de ce groupe (n=21) ont été interrogés de nouveau en posant les mêmes questions sous des formes différentes. Une grande déformation s'est produite par l'utilisation de la stratégie d'addition dans ces autres questions, et on y a attiré l'attention de l'enfant. On a pris note des réactions des enfants interrogés, et également du procédé de solution adopté face au conflit. Ensuite on a poursuivi une étude pilote à la base de l'enseignement de quatre aspects importants à la solution des problèmes d'agrandissement chez quatre groupes d'enfants identifiés par l'emploi de la fausse stratégie d'addition.

Part of the CSMS research on secondary school children's understanding of mathematics was concerned with the topic of Ratio and Proportion (Hart, 1978). From the results of interviewing thirty children and testing 3,000 children aged 12+ to 15+ years with a written test it was possible to formulate certain hypotheses on why certain questions were easy and on what incorrect method a number of children were using on four hard questions. Karplus et al (1975) and Inhelder and Piaget (1958) have identified a common incorrect strategy employed by children when faced with a proportion problem. This strategy (referred to as "the incorrect addition strategy") stemmed from the belief that enlargement could be produced by the addition of an amount rather than by the employment of a multiplicative method. The child would, in this example, reason that since the base line had been increased by two units so must the upright be so increased.

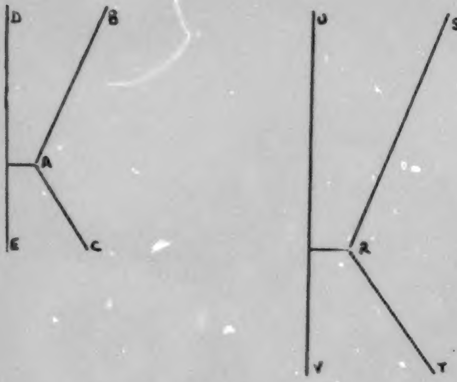


Enlarge with a base of 5

The type of answer that would result from using an incorrect addition strategy was very prevalent on four of the hardest questions on the written test paper (about 40 per cent of each age group). A third of the total sample (n=3000) consistently produced this type of answer on three of the four questions. Three of the four questions concerned enlargement of figures, the fourth was a version of Karplus' "Mr. Short and Mr. Tall".

The new project "Strategies and Errors in Secondary Mathematics" funded by the SSRC proposes to look in depth at this incorrect addition strategy. Phase one of the investigation was designed to select for interview children who produced addition type answers on the four CSMS questions. Consequently five London schools were asked to give an average class of each age group (13, 14, 15 years) the CSMS Ratio test. Forty children so identified were then interviewed on two questions for which they tended to have the correct answer, and on the four 'addition' questions. They completed items requiring 2:1 by halving or doubling, 3:2 by saying 'take it once and take a half and add' and 5:2 similarly. They found it much more difficult to find a smaller amount given the larger in the ratio 5:2 unless a pronounced number pattern was present. The easy questions were therefore successfully completed by using some form of addition. The four hard questions were incorrectly done by the addition of the difference.

The second phase of the research again involved interviewing a subset of these same children and another five 'adders' on alternative forms of the four questions, forms involving the same basic ideas but different numbers, for example:



These 2 letters are the same shape,
one is larger than the other.
AC is 4 units. RT is 6 units.

AB is 7 units. How long is RS?
UV is 15 units. How long is DE?

A crucial difference in this second set of interviews was that whatever answer the child produced, for example for RS (usually 7+2=9), he was given a strip of card of that length and asked to check whether it was correct. Two boys questioned whether the given enlargement was correct and a further three tried to adapt the figures in some way, e.g. "if you pull this line down it will be about the same length as that, so ....".

Two changed their method to a form of addition which was correct (7+½(7)) and two changed their method completely and multiplied by 3/2. Generally there was an acceptance that the addition strategy gave an incorrect answer but there was nothing worthwhile to replace it.

The final aim of SESM is to produce classroom teaching modules, these would in some way prevent or remediate the identified errors. For a pilot study designed to eradicate the incorrect addition strategy in the solution of geometric enlargement problems a number of variables were identified. The interviewed children had all been taught ratio at some time but when faced with the four questions they had

a) not multiplied to obtain an enlargement (or indeed to deal with any ratio question)

b) refused to multiply fractions in any form

c) omitted to find a multiplying factor (scale factor) relationship from the data

d) reacted to the knowledge that their method produced incorrect answers.

It seemed likely that any remedial programme would have to involve a stress on multiplying or 'times' (T), multiplication of fractions (F), scale factor (S) and conflict in the realisation that addition was incorrect (C). It was not certain that all these would be needed but small group teaching would produce evidence of the insufficiency of certain combinations of T,F,S,C.

The next research phase was therefore the teaching of four small groups of children who had been identified as 'adders' and had already received some teaching in their normal lessons on the topic of ratio and proportion. The underlying educational theory of these lessons was that

a) discussion would be encouraged,

b) child-suggestions would be followed up,

c) there would be continuous assessment and immediate feedback,

d) 'child-methods' would be accepted but multiplication of fractions would be the preferred method and children encouraged to move towards this,

e) the teacher would try at all times to say 'times bigger' and at no time to recommend adding.

This form of teaching is based on the implications of the CSMS results, a) children tend to use naive or 'concrete' methods for solving problems long after more formal methods of solution have been taught and the only way to discover these methods is to talk to the children b) there is a wide range of attainment in any age group and it is important to discover the level at which the child is working. Since the 'child-methods' are limited it is important that at some stage the children committed to them be persuaded to turn to a more generalisable method which can be used to solve all problems of a particular type. The correct addition or building-up method used on easy ratio questions (n:2) is abandoned by the child when the computation involved is difficult or a fraction other than one half is involved. Therefore for the solution of other ratio or proportion questions a general method is needed. The transition from one to the other would seem to need specific teaching, it does not come about automatically.

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CONSTRUCT ANALYSIS, MANIPULATIVE AIDS, REPRESENTATIONAL SYSTEMS AND LEARNING RATIONAL NUMBER CONCEPTS*

[An Update on Activities of the Rational Number Project]

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Sommaire :

Une étude de recherche éducative financée par The National Science Foundation of the USA est en cours sous la direction de M. Lesh (Northwestern University) M. Behr (Northern Illinois University) et M. Post (University of Minnesota) dans le cadre de l'enseignement fondé sur la théorie. La base théorique de cette étude comprend a) les aspects de l'analyse de Kieren du nombre rationnel en 5 subconstructions.

b) les principes de Z.P. Dienes sur l'intégration multiple et la variabilité mathématique

c) une extension des 3 modes de Bruner de la pensée représentationnelle

d) la psychologie du traitement de l'information en ce qui concerne le développement des divers genres des structures de la mémoire.

L'Etude du Nombre Rationnel contient également une partie sur l'évaluation. Une méthode d'ensemble portant sur les concepts, rapports et opérations du nombre rationnel est en cours de développement et d'utilisation chez 1600 élèves et plus des grades 2 à 8 (de 7 à 13 ans) dans 5 endroits différents. Des entrevues à fins d'évaluation et des tests éducatifs sont également expérimentés avec des enfants.

Le matériel éducatif développé dans le cadre de cette théorie intégrée est offert à de petits groupes (6 membres) d'enfants âgés de 9 et 10 ans (grade 4 et 5) dans des classes quotidiennes sur une période de 16 à 18 semaines. Une observation régulière et systématique de la pensée et performance des enfants est faite pendant le cours ; ces observations ajoutées aux fréquentes entrevues individuelles avec les enfants, fournissent la majorité des données.

L'analyse de ces données fournira un aperçu des étapes de développement du concept du nombre rationnel à partir du tout début en passant par les phases préliminaires de raisonnement proportionnel.

*Paper Presented at the Fifth Conference of the International Group for the Psychology of Mathematical Education. Grenable France, July 1981.

This paper is based in part on research supported by the National Science Foundation under grant number SEd 79-20591. Any opinions, findings, and conclusions expressed are those of the authors and do not necessarily reflect the views of the National Science Foundation.

INTRODUCTION

The Rational Number Project (RNP) is a cohesive program for research on rational number learning. This program consists of well-defined theory based instructional and evaluation components as well as an overall plan for validating project generated hypotheses. This effort differs along a number of substantive dimensions from previous research efforts in the area of rational number learning.

Previous researchers have focused attention on "state of the art" type research, that is data collection without formal instruction. Kieren (1976), Novillis (1976), Noelting (1978, 1980), Hart (1978, 1981), and Karplus (1980) have utilized paper and pencil tests and interview results to formulate hierarchies of rational number and proportional reasoning concepts. These studies have provided important insights into the hierarchical nature of the acquisition of these concepts. The Michigan Studies (Payne 1976) examined various approaches to fraction algorithm and various manipulative materials over an eight year period of time. The initial comparative studies have evolved into studies more concerned with assessing the quality and durability of evolving cognitive structures. This latter concern is more closely related to our work. In an attempt to extend and reformulate previous efforts, (RNP) has developed and implemented a complete instructional and evaluation program. Our intent is to describe rational number development from its genesis to its formal operational level in well defined instructional settings. The major concern is the identification of psychological and mathematical variables which impede and/or promote the learning of rational number concepts.

THEORETICAL FOUNDATIONS OF THE PROJECT

Of particular concern to the project are three components of learning and knowing concepts of rational number. The first involves a logical mathematical analysis of rational number (Kieren, 1976) and the integration of this mathematical analysis with categories of manipulative aids in the context of theory developed by Z.P. Dienes The second involves an interactive model for describing modes of representation, and the third involves delineation of various memory structures which are developed by the learner as a result of exposure to a theory-based instructional sequence.

I. Kieren (1976) provided a logical analysis of the rational number concept into five subconstructs -- part-whole relationships, measure relationships, ratio, quotient, and operator. Post (1974) and Post and Reys (1979) have integrated Kieren's work with a logical analysis of concrete models for representing rational number concepts. This model, presented in Figure 1, incorporates the mathematical and perceptual variability principles of Dienes (1967). This analysis provides an organizational scheme for the development and selection of appropriate instructional materials.

Consideration of both rows and columns provides for both the abstraction and generalization of these concepts.

		MATHEMATICAL VARIABILITY				
		PART WHOLE	MEASURE	RATIO	QUOTIENT	OPERATOR
PERCEPTUAL VARIABILITY	COUNTERS: SET-SUBSET INTERPRETATION					
	PAPER FOLDING: AREA INTERPRETATION					
	SYMBOLIC ALGORITHM(S)					

FIGURE 1

OPERATIONAL DEFINITION OF THE CONCEPT OF RATIONAL NUMBER FROM WHICH INSTRUCTIONAL ROUTINES ARE DEVELOPED

Based on this matrix we have conceptualized instruction for rational numbers as suggested in Figure 2.

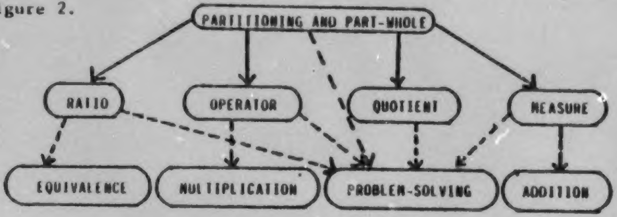

FIGURE 2
CONCEPTUAL SCHEME FOR INSTRUCTION ON RATIONAL NUMBERS

The arrows and dashed arrows in Figure 2 are to suggest hypothesized relationships between rational number concepts, relations, and operations. The diagram suggests that (1) partitioning and the part-whole construct of rational numbers are basic to learning other constructs of rational number; (2) the ratio construct is most "natural" to maturate the concept of equivalence and nonequivalence; (3) operator and measure constructs lend themselves to the understanding of operations of multiplication and addition.

II. The Modes of Representation and translations emphasized in the project materials are depicted below. The reader will note that Figure 3 represents an extention of Bruner's early work on representational modes. Lesh (1979) reconceptualized Bruner's (1966) enactive mode, partitioned Bruner's iconic mode into manipulative materials and static figural models (i.e., pictures), and partitioned Bruner's symbolic mode into spoken language and written symbols. Furthermore, these systems of representation were interpreted as interactive rather than linear. The revised model follows:

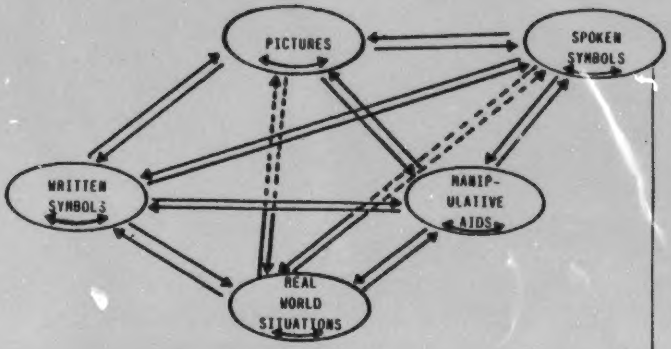

FIGURE 3
An Interactive Model for Using Representational Systems

A major hypothesis of the project is that it is the ability to make translations among and between these several modes of representation that make ideas meaningful to learners.

This interactive instructional model for modes of representation is one that needs refinement through empirical verification to determine which of the many translations are crucial in mathematics learning. Two triads in the model are of particular interest in our research. One involves the translations between manipulative aids and mathematical symbols, with the oral mode serving as a mediating facilitator in this translation process. The other involves real world situations, manipulative aids, and written symbols; of concern is the question of how to use manipulative aids to facilitate the mathematical modeling required in problem solving.

III. Various writers discuss categorizations of memory. Gagne and White (1978) consider the relationship between memory structures and learner performance. Of interest to this project are memory structures called episodic, imaginal, semantic, and intellectual skills.

COMPATIBILITY OF THE THEORIES

Figure 1 suggests that learning will be enhanced when overt attention is paid to the nature and scope of both the manipulative materials and the mathematical dimensions of the concept to be learned. Figure 3 predicts that mathematical learning, retention, and transfer will be enhanced when instructional routines provide for interaction among and within the various modes of representation. The memory related literature suggests that learning, retention, and transfer will be greater when interrelationships among memory structures are made. The interactive model suggests instructional variables for investigation in order to determine their effects on a learner's thinking processes, and the memory literature directs a researcher's attention to the observation of behaviors that suggest the existence of specific thinking processes. Thus, the three theories are not only compatible but also provide a powerful framework for investigating the phenomena involved in learning from manipulative materials.

THE RATIONAL NUMBER PROJECT

The RNP has three distinct yet complementary components: Instructional, Evaluation, and Diagnostic and Intervention. All adhere to the same theoretical and philosophical foundations. Twenty weeks of student instructional materials have been developed. The instructional materials reflect the project's underlying theoretical foundations and emphasize part-whole, quotient, measure, and ratio interpretations of rational number, and involve translations within and between five representational modes. (see Figure 3).

INSTRUCTION

Instructional activities with children began in mid-October, 1980 and continued thru March, 1981. Three groups of 6 children (4th grade in DeKalb, 4th and 5th grade students in Minneapolis), were instructed daily using theory based project generated materials. These materials addressed many of the standard rational number concepts, but in addition paid particular attention to the use of manipulative aids and translations within and between various modes of representation. Extensive observational data were taken during and immediately after instruction, much of which was recorded on video tape for subsequent analysis. A minimum of three persons were present at each of these instructional sessions. (One teacher and 2 observers)

DATA COLLECTION AND ANALYSIS

Observational data, frequent interviews, and audio or video taping of many lessons resulted in a large amount of anecdotal data. In addition, four major types of instruments have been employed by the instructional component.

1. The Rational Number Test - identified levels of student achievement in three areas: rational number concepts, relations and operations. These tests which were developed by the projects evaluation component were used with project instructed children and with classroom sized groups in grades 2 thru 8 (ages 7-12) across give geographic locations (N > 1600).
2. Class observation guides were designed to provide insights into the cognitive processes employed by students when dealing with rational number concepts in the structured instructional setting.

3. Interview Protocols. The individual interview, conducted with each student after each lesson, is considered a crucial source of project data. These interviews, lasting from 15 to 50 minutes, provide extensive information as to the mental processes, memory structures (inferred), thought patterns, and understandings gained and utilized. Interview data is examined on a lesson-by-lesson basis to assess the impact of specific instructional "moves" on conceptual development. Either audio or video tapes were always used to provide a record of these interviews.

4. Translation Coding System. This instrument was designed to provide specific information as to the types of translations which students used, the relative frequency of each type, and the identification of those which proved particularly troublesome.

In addition to these instruction related instruments, the evaluation component has also developed a series of clinical interviews and instruction mediated tests. Together the data gathered with these instruments will provide a rather comprehensive view of rational number development in children and should add substantially to the body of knowledge already in existence.

Our work has led to the following observations about the use of manipulative aids. Each is supported by extensive observation, anecdotal records and audio or video tapes:

1. Use of multiple aids to represent a concept is more helpful in children's learning than use of a single aid.
2. After a concept is initially introduced with a chosen manipulative aid, subsequent representations with manipulative aids which differ in perceptual features cause the child to rethink the concept and learning is facilitated.
3. A method for introducing a "new" manipulative to the discussion of a given concept has been devised, tested, and proven successful.
4. In order for a manipulative aid to facilitate learning, it appears necessary that it initially cause cognitive disequilibrium. We believe this to be in striking contrast to what one gleans from current mathematics education literature.

Space constraints here preclude consideration of project results in any detail. The presentation at Grenoble will focus specifically upon our findings related to 1) translations within and between modes of representation and 2) the impact of perceptual distractors on the quality of childrens rational number thinking. A second paper, providing more details, will be distributed at that time.

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COMPUTATIONAL ERROR OF SEVENTH AND EIGHTH GRADE STUDENTS

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ABSTRACT

Cette étude avait pour but de reproduire et d'étendre les efforts de Roberts et les interviews cliniques d'Engelhardt visant à la classification et à l'analyse, respectivement, des erreurs de calcul des enfants. Chaque jour pendant cinq jours consécutifs, cinq groupes différents d'élèves en 6^e et en 5^e ont dû passer une interrogation écrite de cinq minutes comportant cinq problèmes de calcul fractionnaire. Après la notation de chacune de ces interrogations -- notation qui tenait compte de résponses exactes, de résponses inexactes, et de non-réponses -- on a interviewé ceux des élèves qui avaient mis des réponses inexactes ou des non-réponses, afin d'arriver à des conclusions sur les approches ou les idées erronées qui auraient conduit les élèves à ces erreurs. Cette méthode a abouti à la création de neuf catégories pour les réponses inexactes ou les non-réponses, dont deux correspondaient à des types d'erreur décrits par Roberts et trois à des types d'erreur décrits par Engelhardt. Ces catégories étaient les suivantes: algorithme défectueus, fractions équivalentes, conversion incorrecte d'un nombre fractionnaire en expression fractionnaire ou d'une expression fractionnaire en nombre fractionnaire, simplification, opération erronée, principe de base, réponse incomplète, non-réponse, et problème mal copié. Après l'examen de ces neuf catégories, on a procédé à des généralisations provisoires.

INTRODUCTION

The goal of this research was to replicate and extend Roberts' (1968) efforts of classification and Engelhardt's (1977) analysis of computational errors exhibited by students. Replication was sought in the sense of analyzing and classifying students' computational errors. Several extensions were made. One was to use fractions instead of whole numbers as Roberts' and Engelhardt's had done. Another extension was to consider not only the incorrect responses but also those which were left blank. A third extension was to interview those students who had the incorrect response or no response. Interviewing the students would allow the classification to be based on how the student derived the answer as opposed to how the investigator thought the student derived the answer. The last extension was to time each of the five tests.

It was hoped that a list could be developed which specified the errors that students made with fractions as well as whether any generalizations could be derived about their attack skills which led to an incorrect response or no response.

METHOD

Sample

The subjects for this study were seventh and eighth grade students from a small city in Texas. There were 105 students in the five classes. Due to absenteeism, 28 subjects were not interviewed for each of the five tests. Of the five classes, only one was homogeneously grouped for mathematics instruction. The sample was divided into quartiles which were defined using each student's percentile on the Iowa Test of Basic Skills.

Procedure

Each student was administered a five item test on fractions for five consecutive days at the beginning of the class period by the investigator. After each five minute test, the papers were collected and graded by the investigator. Each student who had either incorrectly responded or no response was interviewed by the investigator. The investigator wrote the student's comments adjacent to the problem(s) that the student had not answered correctly or not answered. These interviews ranged from one minute to five minutes, depending on the number of errors, per student for each of the five days.

After the students had been interviewed, the incorrect items and no response items were identified and analyzed to determine classes of errors on the basis of the interviews. In several of the incorrect responses, the students had made several error types which were classified accordingly to error type. The distribution of error by class types among the sample was examined for possible generalizations.

Instruments

The tests used in this study were constructed by the investigator. The five tests---1 Addition, 2 Subtraction, 3 Multiplication, 4 Division, and 5 Working With Fractions---were each hierarchically designed (Uprichard and Phillips, 1977) so that there was an increase of difficulty from one problem to the next.

RESULTS

Types of Errors

It was possible to establish nine major classes of incorrect responses or no response on the basis of the interviews and the analysis. These were:

(1) Defective Algorithm: The student attempts to apply the appropriate operation but makes errors in carrying through the necessary steps. Examples were:

(a) $1/2 + 1/3 = 2/5$ (b) $1/7 \times 2/7 = 2/7$

(c) $3\text{-}4/5 \times 2\text{-}7/9 = 6\text{-}28/45$ (d) $1/6 \div 5/7 = 6/1 \times 5/7 = 30/7 = 4\text{-}2/7$

(2) Equivalent fractions: The student attempts to rename a given fraction but fails to complete the necessary steps. This error type included renaming for borrowing or carrying. Examples were:

(a) $6\text{-}3/4 + 1\text{-}7/8 = 6\text{-}2/8 + 1\text{-}7/8 = 7\text{-}9/8 = 8\text{-}1/8$

(b) $3/4 - 1/3 = 6/12 - 4/12 = 1/6$

(3) Mixed numeral to improper fraction or improper fraction to mixed numeral: The pupil attempts to rename a mixed numeral as an improper fraction or an improper fraction as a mixed numeral and makes a computational error in completing the task. One example was

$4\text{-}1/6 + 9\text{-}3/5 = 25/6 + 48/5 = 125/30 + 288/30 = 413/30 = 13\text{-}19/25$

(4) Simplifying: The student attempts to reduce a fraction to its lowest terms or rename the mixed numeral. Examples were:

$7/8 \times 1/7 = 7/56$ $4\text{-}4/7 + 3\text{-}5/7 = 7\text{-}9/7$

(5) Wrong Operation: The student attempts to respond by performing the incorrect operation than the one required to solve the problem. An example was $1/7 \times 2/7 = 3/7$

(6) Basic Fact: The pupil responds with a computation involving an error in recalling basic number facts. Examples were:

(a) $12\text{-}3/4 - 2\text{-}5/8 = 12\text{-}6/8 - 2\text{-}5/8 = 10\text{-}3/8$

(b) $7/8 \times 1/7 = 7/53$

(7) Incomplete or Guess : The student attempts to solve the problem but either he/she does not complete it or guesses. Examples were:

(a) 1/2 + 1/3 = 3/6 x 2/6 = 2/5

(b) 7/8 x 1/7 = 343/40 = 7

(c) 3-4/5 x 2-7/9 = 15-5/9

(d) 4-1/6 = /30
9-3/5 = /30

(8) Did Not Attempt: The student did not start any computation on the problem.

(9) Miscopied: The student incorrectly rewrites the problem. Some examples were:

(a) 4-1/6 + 9-3/5
4-1/2 = 5/10
+ 9-3/5 = 6/10
13-11/10 = 14-1/10

(b) 1-1/6 ÷ 7 = 7/1 x 1/7 = 7/7 = 1

(c) 6-3/4 + 1-7/8 = 63/4 + 17/8 = 126/8 + 34/8 = 160/8 = 20

It should be noted that the defective algorithm and the wrong operation correspond to Roberts', and Engelhardt's 'inappropriate inversion', 'defective algorithm,' and 'incomplete algorithm' correspond to this study's defective algorithm.

Distribution of Errors

The 77 students in the sample committed 736 errors with an average of 23.7 problems attempted of the possible 25 and with a correct response to 16.4, thus committing 8.3 incorrect responses and 1.3 no response. From quartile high to low shows the number of attempted problems to decrease in most instances, and the number of correct responses to decrease. Although students in the lower quartile attempted fewer problems, they committed more errors.

TABLE 1

Computation Performance by Quartile

	High	Medium High	Medium Low	Low	Total
	N = 19	N = 120	N = 17	N = 17	
	No. Mean	No. Mean	No. Mean	No. Mean	
Items Attempted	469 (24.7)	571 (24.8)	404 (23.8)	382 (22.5)	1826 (23.7)
Items Correct	395 (20.8)	421 (17.5)	256 (15.1)	189 (11.1)	1261 (16.4)
Incorrect Responses	77 (4.1)	162 (6.8)	167 (9.8)	232 (13.6)	638 (8.3)
No Response	5 (.3)	29 (1.2)	21 (1.2)	43 (2.5)	98 (1.3)

The number of errors for each error type and the percentage of each error type is presented in Table 2.

Table 2

Distribution of Errors by Type

	Quartile				
Error Type	High	Medium High	Medium Low	Low	Total
Defective Algorithm	27	50	73	112	262 36%
Equivalent Fractions	5	20	9	18	52 7%
Mixed Numeral to Improper Fraction or Improper Fraction to Mixed Numeral	9	15	17	11	52 7%
Simplifying	4	8	11	14	37 5%
Wrong Operation	4	10	11	20	45 6%
Basic Fact	7	15	10	11	43 6%
Incomplete or Guess	12	30	29	42	113 15%
Non-attempt	5	29	21	43	98 13%
Miscopied	9	14	7	4	34 5%

As is evident from the totals, defective algorithm, incomplete, and no response were the most common types of errors. About one-seventh (100) of the total number of errors were the result of classifying errors as two or more errors type.

DISCUSSION

It is apparent from Table 2 that over 50% of the errors were due to defective algorithm, incomplete or guess, and non-attempt. This suggests that in teaching the various algorithms one should stress how one algorithm differs from another so that transfer will not take place incorrectly nor incompletely. One example was the student who did 1/7 x 2/7 = 2/7; another was 6/7 ÷ 7 = 7/6 x 7/1 = 49/6 = 8-1/6.

The incomplete responses or guesses and the non-attempts were equally caused by the time element or the students not knowing what to do.

Additional studies need to be conducted in which responses are based upon clinical interviews as this one with different numbers as well as subdividing the defective algorithm to include incomplete altorithm and inappropriate inversions as Engelhardt's study had. Future studies should not include the time element. It can be a factor in the number of errors, and possibly the type of error, students exhibit.

A replication and extension of this study is needed in that the results of past studies, this study, and future studies will be an aid for teachers in identifying computational difficulties of students for different types of numbers.

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DOCUMENTATION OF TEACHER MOVEMENT TOWARD PROCESS-ENRICHED TEACHING OF FRACTIONS AND RATIOS*

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Le Calgary Junior High School Mathematics Project *a été conçu pour combler l'écart éxistant entre les niveaux de développement cognitif des etudiants de 12 à 14 ans et les exigences cognitives de leurs cours sur les fractions et rapports. Sept professeurs qui auraient accepté de participer à l'expérience ont été familiarisés pendant cinq jours avec une variété de matériel didactique centré sur le processus(enrichi et inspiré du matériel didactique du* South Nottinghamshire Project). *On leur aussi demandé de participer à la production du matériel centré sur le processus concernant des fractions et des rapports. L'effet que les cinq jours d'orientation et de préparation ont eu sur la didactique des professeurs a été établi par des observations de classes normales et de classes expérimentales et par des questionnaires (professeur et étudiant). Nous mentionnons brièvement les progrès significativement plus importants réalisés dans les classes expérimentales en ce qui concerne l'attitude, les résultats obtenus et les "stratégies mathématiques générales." Cet article met l'accent sur les descriptions qualitatives et quantitatives de l'habilité des étudiants, des exigences du programme, de l'emploi du matériel didactique centré sur le processus, et des contrastes entre les méthodes d'enseignement observées dans les classes expérimentales et les classes régulières. (Traduit par J. Paquet.)*

In the contexts of "ratio and proportion" and "rational numbers" at the grade seven and eight levels, the purpose of the Calgary Junior High School Mathematics Project (CJHMP) was to assess: student cognitive abilities, curriculum guide cognitive demands, classroom cognitive demands, and the effects of using a process-enriched instructional method which allows for differing student cognitive abilities. The cognitive ability and demand levels used in the study were defined as follows:

> Concrete, refers either to the ability to handle fraction situations involving: taking one-half or one-third of a set of objects, a drawing, or a number; doubling or tripling to produce equivalent fractions; or adding or subtracting with physical models; or to the

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*Extracts from the Final Report of the Calgary Junior High School Mathematics Project, December 1980, jointly funded by Planning and Research, Alberta Education, and the Calgary Board of Education.

ability to handle ratio and proportion situations involving: 1 to 2, 1 to 3, or their reciprocals; only doubling or tripling or the reverse; or action with countable real objects.

Transitional refers to the ability to handle fraction or ratio and proportion situations involving: pictorial support containing countable parts, or abilities beyond the concrete but not formal.

Formal refers either to the ability to handle fraction situations involving: at least one element greater than 4; comparisons made in terms of different sized units; second order thinking; symbolic representations of fractional relationships and operations; or expectation of a correct explanation of procedures; or to the ability to handle ratio and proportion situations involving: at least one component greater than 4; second order proportional thinking; symbolic representations of proportional relationships; or explanation using proportional reasoning.

PROCEDURES AND FINDINGS

Cognitive Abilities versus Demands

In one phase of the CJHMP, assessments were made of student cognitive levels in grade seven and eight fraction and ratio contexts and of the corresponding cognitive demands made by curriculum guide objectives, authorized textbooks, teacher presentations and teacher-made tests. Of the 435 grade seven students tested in the context of fractions, 55.0 percent were at the concrete level, 38.2 percent were transitional, and only 6.0 percent were formal. For grade eight (312 students), the figures were 44.5 percent concrete, 43.3 percent transitional, and 12.2 percent formal. In the treatment of fractions by regular textbooks, teachers and tests in the two grades, the cognitive demands were found to conform to the concrete level less than 4 percent of the time and to the formal level more than 68 percent of the time. Similar patterns were found for ratios. These findings indicated a considerable gap between student cognitive level and curricular demand in the contexts of grade seven and eight fractions and ratios.

Concrete Process-Oriented versus Regular Teaching

Another phase of the CJHMP analyzed the effects of a concrete process-oriented (CPO) approach to the teaching of fractions and ratios in seventh and eighth grades as compared with the regular (REG) approach typical of North American schools. The effects measured included achievement in fractions and ratios as well as attitude and ability to apply general mathematical strategies. Six teachers taught 246 grade seven students in ten experimental classes while 234 grade seven students in nine classes were taught by four teachers using regular classroom teaching methods. At the grade eight level, 120 students in four experimental classes were taught by two teachers while

213 students in eight regular classes were taught by four teachers. The students had been matched on standardized school-administered tests. All 813 students were pre-tested on achievement in fractions and ratios, attitude toward fractions and ratios, and general mathematical strategies. Both groups of grade seven students were taught fractions and ratios for approximately twelve weeks. For the grade eight groups, the length of treatment was approximately eight weeks. At the completion of each unit all of the treatment groups were retested using the same achievement and attitude instruments as those administered initially.

The experimental teaching methods used in the project were modelled on the pioneering work of the South Nottinghamshire Project (SNP). The SNP teaching approaches characteristically use a variety of simple concrete materials to pose "well-motivated" problems in which the concrete "props" facilitate understanding of the mathematical ideas involved (Bell, 1976, p. 5.4). A pupil carries out a mathematical investigation beginning with concrete materials, experimenting, recording what happens, formulating questions, and writing-up accounts of experimental results as well as applying the results to practical situations. Such strategies are not generally embodied in North American school mathematics textbook materials nor do they occur with any frequency in the most common secondary school mathematics mode of instruction: "example-rule-exercises."

Towards the end of August, 1979, the experimental teachers were presented with process-enriched material from various sources which covered the topics in fractions and ratios as outlined by the Alberta Education Curriculum Guide for grades seven and eight. Several investigations were tried out with the experimental teachers, teaching strategies were discussed and teacher decisions were made regarding adapting the materials to their classrooms. From October through December, when the classroom treatments were in progress, periodic classroom observations were made to determine whether or not the experimental teachers actually did use teaching methods that were demonstrably different from those used by the regular teachers. An observer or, on occasion, two observers, would sit quietly at the back of the class being observed and would record codes and/or comments in relation to one-minute intervals of elapsed class time. The codes indicated what kind of grouping, teaching method, materials, teacher questions, and cognitive demands were observed, and the positioning of the codes on the sheet indicated how long they lasted. Highlights from the overall patterns observed are given in Table I.

Another kind of record was also kept. Subsequent to an observed lesson, a short written summary of the lesson content was made along with Likert-scale ratings of the frequency with which twenty-one different classroom behaviours were observed. For example, the use of real objects when learning new ideas and while investigating fraction and ratio problems occurred with moderately high frequency in the experimental classes but was not observed in the regular classes (4.4 to 0.0 on a Likert frequency of observation scale with 0.0 for low and 5.0 high.) Similarly, student writing of reports resulting from small group investigations of fraction and ratio problems was observed with moderately high frequency in the experimental classes but was not observed in the regular classes (3.8 vs 0.0). The experimental teachers demonstrated a moderately high frequency of relating independent activities and investigations to concepts being learned as well as providing for a gradual transition from concrete to more abstract activities; the frequency of such activities in the regular classrooms was low (3.8 vs 1.5). In the regular classrooms there was a moderately high frequency of students being given rules and/or examples before problems were attempted or new work started, while such observations had a low frequency in experimental classes (3.9 vs 1.5). Both regular and experimental teachers displayed similar frequency patterns in using systems for spot-checking student assignments and in providing evidence of caring, accepting, and valuing student responses (range: 3.2 to 4.0).

TABLE 1

SUMMARY OF CJHMP CLASSROOM OBSERVATION FINDINGS
AVERAGE PERCENTAGE OF CLASS TIME SPENT IN EACH MODE

GROUPING	Individuals	Pairs	Small Groups	Whole Class
Experimental	35%	18%	8%	39%
Regular	48%	0%	0%	52%

METHOD	Dialog	Expos	Invest	Real Ob	Discus	Text Ex	Cor Work	Other
Experiment	19%	8%	4%	31%	2%	7%	21%	7%
Regular	8%	26%	0%	0%	7%	49%	6%	4%

MATERIALS	Concrete	Pictorial	Demo	Text	Worksheets	Chalkbd	Unclassif
Experimental	31%	7%(man.)	7%	18%	15%	3%	18%
Regular	0%	18%(demo)	13%	48%	21%	0%	0%

COGNITIVE DEMAND*	CON	TRA	FOR
Experimental	15%	68%	17%
Regular	0%	51%	49%

CON: concrete
TRA: transitional
FOR: formal

*Percentage expressed in terms of total time teacher made cognitive demands; averaged across fractions and ratios.

Dialog: Dialogue; Expos: Exposition; Invest: Investigation
Real Ob: Real Objects; Discus: Discussion; Text Ex: Text Exercises;
Cor Work: Correcting Work; Chalkbd: Chalkboard; Unclassif: Unclassified.

Analyses of the student questionnaires that were administered indicated a significant difference between the experimental and regular class teaching methods as perceived by the students. One question, for example, which referred to being told a rule before attempting a problem, showed a significant difference between experimental and regular classes, with this happening much less frequently in the experimental classes. The analyses also showed that students in the experimental classes more frequently used manipulatives for investigating ideas in fractions and ratios. They were also more frequently encouraged to invent their own solutions to problems, to notice number patterns, to work in pairs or groups, and to write reports of what they had found out while investigating fraction and ratio problems.

Statistical Findings

Two-factor repeated measures analyses of variance were used to compare treatment groups on attitude towards fractions and ratios (anxiety and enjoyment) as well as the ability to apply general mathematical strategies. Significant differences were found at the 0.05 level favouring the grade seven experimental (CPO) group on all of these measures. However, at the grade eight level, no significant differences were found between the attitude scores or overall general mathematical strategy scores of the two treatment groups.

The CSMS (see Hart, 1981) fraction and ratio achievement test scores were analyzed using three-factor repeated measures analyses of variance, with treatment group, cognitive level, and test occasion (repeated) as factors. The results of these analyses favoured the CPO group with a level of significance of 0.05. There were no significant differences between the treatment groups in "computation with fractions" scores at either grade level. *A priori* comparisons using t-tests at each of the grade seven cognitive levels indicated that there was a significant difference, favouring the CPO group, between the mean increases in fraction scores attained by the CPO and REG *transitional* level students, but no significant differences were found at the *concrete* or *formal* levels. The grade eight t-tests indicated a significant fraction score difference at the *formal* level favouring the CPO group. The t-test analysis of the grade seven ratio test score increases showed significant differences at the 0.05 level favouring the CPO group at both the *transitional* and *formal* levels, but no significant differences were found at the grade eight level.

That the CPO groups generally achieved higher scores than their REG counterparts on measures of achievement and attitude might well be attributable to the attention paid in the CPO method to the subconstructs of rational number

(Kieren, 1976), to the use of concrete materials and investigations (Bell, 1976), and to the conscious provision for student reflection and discussion to promote relational learning (Skemp, 1979).

CONCLUSIONS

A substantial gap between student cognitive levels and curricular demands has been documented in the contexts of fractions and ratios with typical grade seven and eight Calgary students.

Overall, but particularly at the grade seven level, the project findings have demonstrated that a concrete, process-oriented approach can result in significantly improved achievement in, and attitude towards, fractions and ratios while enhancing the development of general mathematical strategies and maintaining computational facility.

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THE PERCEPTION OF RANDOMNESS

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Le concept de hasard joue un role important dans de nombreux domaines de la connaissance et de la pensée. Realiser qu'un phenomène n'est pas fortruit indique souvent que l'on doive en expliquer la cause ou établir une theorie. Ceci est particuliérment frappant en science, où un "resultat significatif" suscite generalement une interpretation autre que le hasard.

Le caractère apparemment fortruit de suites binaires de stimuli a été étudié comme une fonction de la structure statistique des suites. On a demandé à des sujets de juger du degré de hasard de certaines suites binaires et tables bi-dimensionnelles, limitant cette étude à des frequences égales des deux symboles. Le principal variable independant était le degré de dependance de premier ordre entre les symboles. Les suites ainsi que les tables variaient des alternations parfaites à un groupement extrême.On a également demandé aux sujets de créer des tables et des suites binaires arbitraires.

Les resultats denotent une deformation de l'image du hasard chez les sujets. Les suites et les tables, presentant une augmentation deliberée d'alternations, furent generalement jugées comme étant dues au hasard, tandis que les continuités d'une et de deux dimensions furent perçues comme non-hasard. "L'erreur du joueur" c'est à dire, la tendance à identifier le hasard à trop d'alternations, dominent à la fois la perception et la creation.

En conclusion, on tend à rejeter trop promptement l'hypothèse du hasard et donc a interpreter à l'exces le mounde qui hous entoure. De même,notre "niveau subjective de signification" est incoherent vu que les excès d'alternations et de groupement de probabilité symétrique furent jugés differemment.

The concept of chance plays an important role in many fields of scholarship and thought. The outset for the intellectual operations of explaining and constructing theories seems to be the realization that the phenomenon in question is nonchance. Every explanation suggests some kind of lawfulness, causality, organization or pattern. Each of the underlined words in the last sentence is an opposite of randomness. This distinction between chance and lawfulness is especially salient in science, where induction may be presented as the process of distinguishing pattern from noise, order from disorder (Lopes, 1980). Typically, a "significant result" calls for some interpretation other than chance. The same course of thought is often pursued in daily life

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1) Parts of this paper present experiments from the author's Ph.D. thesis (Falk,1975), that was supervised by A. Tversky. The study was partly supported by The Human Development Center, the Hebrew University of Jerusalem.

and social affairs, as well as in literature.

A nonchance perception is a cognition that may well affect our behavior. Perceiving a situation as lawful encourages a skill rather than a chance orientation. A systematic phenomenon calls for systematic behavior. One may try to control the situation by replicating or changing it, and eventually by avoiding it. On the other hand, there seems to be no point in patterning our behavior in a totally random environment.

The perception of a situation as more or less random thus seems to be the key to important cognitions and behaviors. A host of studies, starting from the early fifties, were touching the topic of the perception of randomness. It was generally claimed that subjects cannot perceive random sets of stimuli as such. Cohen (1960) summarizes a series of experiments conducted with children and adults with the following contention: "Nothing is so alien to the human mind as the idea of randomness." Cohen's experiments presented mainly probability-learning tasks (i.e., sequential prediction of binary random events) and so were many other studies from which the conclusions about the inability of subjects to perceive randomness were derived. Most of the sequences produced by subjects, as their predictions or guesses, deviated seriously from randomness. The dominant kind of violation of randomness was producing too many alternations between symbols, better known as "The Gambler's Fallacy". This was true for tasks of generation of randomness as well (Wagenaar, 1970a, 1972). The claim that subjects' deviations result from their distorted image of randomness may be true; however, there is an inferential leap in drawing that conclusion. Failure to perceive randomness is but one possible explanation. The sequences produced in probability-learning experiments may be influenced also by the subjects' own previous responses and by their sequence of reinforcements. The response sequences may reflect the hypotheses, concerning the nature of the experiment, entertained by the subjects (Peterson, 1980) and the strategies they developed to cope with the problem solving situation as they interpreted it. Even sequences generated under direct instructions of randomness are not necessarily a mirror reflection of the subject's perception of randomness (Wagenaar, 1972). It is conceivable that a person would be able to perceive randomness accurately and wouldn't be able to replicate it. A direct task of perception of randomness would be necessary in order to establish deviations from randomness such as the gambler's fallacy as perceptual distortions (Wagenaar, 1970b).

The prevalent claim, described above, about subjects' nonacceptance of randomness appears to be an overgeneraliztion. Rather than ask whether subjects do or do not reject randomness, we should better study the factors that determine how random a siuation is perceived. It seems essential to establish the conditions under which randomness is too readily rejected, and, conversely, to try to delineate eventual nonrandom sets of circumstances that might induce a perception of chance.

The aim of the present study is to investigate the apparent randomness of stimuli as a function of their statistical structure. I propose to manipulate the probabilistic characteristics of sets of stimuli and to study the effect of these manipulations on the dependent variable of perception of randomness. Most of the studies in this area were limited to stimuli presented in a sequential format. It would be desirable to extend the organization of the stimuli into more complex patterns so as to get closer to "real-life" situations. Two dimensional tables of stimuli will thus be studied along with one dimensional sequences. The subjects' task will be to give an immediate judgment as to how random the stimuli appear, so that perception would not have to be inferred.

THE EXPERIMENTS

General design: Sequences (in one dimension) and tables (in two dimensions) of binary symbols were designed to be presented for perceptual judgment of their degree of randomness. Likewise, comparable tasks of generation of randomness in one and two dimensions were designed. Parallel stimuli in perception and generation tasks may help to determine whether patterns that appear in the generated sets are due to response tendencies or to perception.

The design was limited to binary sets with equal frequencies of the two symbols. The major independent variable in the perception tasks was the degree of first order statistical dependency characterizing each set. The same variable should serve as a dependent variable in the analysis of the productions obtained under instructions to generate random sets. First order dependency refers to the conditional probability of a symbol given the value of its preceding one. Our binary "symbols" assumed the form of two different colors. This way it was the probability of change (or alternation) of color between successive units that was controlled. The sets included sequences and tables with independence between successive symbols as well as with increased and decreased tendency to alternate, relative to randomness.

Four kinds of experiments resulted from combining perception and generation tasks with the dimensionality of the stimuli-set.

The Perception Experiments: One dimension. Two alternative sets, each consisting of 10 sequences, were prepared for judgment of randomness. Each sequence was composed of 21 cards, 10 green and 11 yellow, or vice versa. The conditional probability of a green (yellow) color given the previous one was yellow (green), i.e., the sequence's probability of alternation, denoted P(AL), assumed the values 0.1, 0.2,...,1.0 in the ten sequences. Except for the constraints concerning the total frequencies, and the required P(AL), all the other features of the sequence were determined by random numbers.

The following are examples of two of the sequences that were presented, Y denotes a yellow card, and G - a green one.

$P(AL)=0.3$: Y Y Y G G Y Y G G Y Y Y G G G G G G Y Y Y $\quad r=7$

$P(AL)=0.8$: G Y G Y G Y G Y Y G Y G Y Y G Y G G Y G G $\quad r=17$

$$P(AL)=\frac{r-1}{20}$$

where r is the number of *runs* in a sequence. Considering r is equivalent to characterizing the sequence by its probability of alternation, since these two statistics are linearly related. The general formula is: $P(AL)=\frac{r-1}{N-1}$, where N denotes the total number of elements in the sequence and r-1 is the number of times color changed.

Two dimensions. 46 tables of 10x10, each comprising 50 green cells and 50 red ones, were constructed. These were divided into 4 equivalent sets of tables, 12 in each of two sets and 11 in the two others.

The concept of run in a sequence (and consequently the statistic r) is not easily extended to a two dimensional table. However, r-1, the number of changes of color in a sequence, can easily be extended to a statistic, denoted k, that counts the number of changes of color upon moving to a neighboring cell either vertically or horizontally. The total number of internal sides of cells in a table of 10x10 is 180. This is also the number of opportunities for color change. Hence, $P(AL)=\frac{k}{180}$ is a statistic measuring the probability of change of color for a randomly chosen transition in the table. The tables, in each set, varied over a wide range of probabilities of alternation, from a near perfect chessboard pattern to very exaggerated clustering. The exact method of the tables' construction is described elsewhere (Falk, 1975).

The task. In both kinds of the perception experiments, the subjects were asked to rate their immediate impression as to how random (well shuffled) such a sequence or table was, on a scale from 1 to 20 (20 - most random).

The Generation Experiments: One dimension. The experiment was run with each subject individually. Subjects were given two decks, one consisting of 20 green cards and the other of 20 yellow ones. They were instructed to arrange the 40 cards in one row the way they would be arranged were they well shuffled.

Two dimensions. Each subject got a table of 10x10, i.e., of 100 empty cells. The subject was instructed to fill 50 of the cells with x-s in a random way.

The Subjects: Most of the subjects were students and graduates of the Hebrew University. Some school children (mostly secondary school) were also included. The total number of subjects amounted to several hundreds.

RESULTS

Perception: For each value of probability of alternation, subjects' ratings were averaged. Figure 1 presents these mean ratings as a function of P(AL), both for one and for two dimensional stimuli.

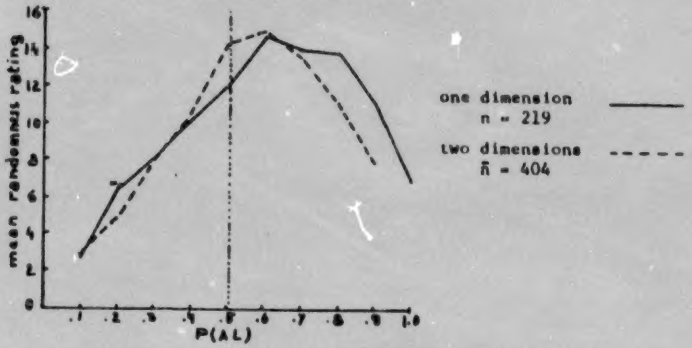

Figure 1. Perception: Mean rating of randomness as a function of probability of alternation. n denotes number of subjects. In two dimensions, different numbers of subjects were exposed to different P(AL)s. n̄ is their mean.

The maximum rating of randomness was obtained for P(AL)=0.6. Were the ratings perfectly correlated with the probabilities of the random variable P(AL) (or alternatively with those of r in one dimension, and k in two dimensions) the maximum should correspond to 0.5. (The sampling distribution of r is given in Siegel, 1956, and that of k is developed in Falk, 1975). Furthermore, the normative judgment function should descend from a point above 0.5 in a nearly symmetrical way to both sides. Thus, the gambler's fallacy in perception is exhibited not only by the fact that the peak of apparent randomness is over 0.6, but also by the negative skewness of both judgment functions (Figure 1). Exact probabilities for the number of runs in short sequences can be computed. It follows, for example, that for a binary random sequence of 10 and 11 symbols of the two kinds, P(AL) of 0.4 is more probable than that of 0.7 and naturally also of any higher value. Note that according to the judgment function for one dimension, not only does P(AL) of 0.7 appear more random than 0.4, but so do also the values 0.8 and 0.9.

The judgment function in two dimensions can be compared to the distribution of 150 random tables that were constructed by using random numbers to fill ("without replacement") 50 cells out of 100 so that each selection was independent of the other ones. The distribution of these tables according to their P(AL) is presented in Figure 2. The value of P(AL) that was perceived more random than any other value, i.e., 0.6 (Figure 1), is equal to the 99-th percentile in the mathematical sampling distribution of random binary tables of 10x10 with 50 cells of each kind. One could thus conclude that a strong effect of identifying exaggerated alternations with randomness, was operating in perception.

Generation: The productions of subjects were analysed as follows: The number of runs, r, for each sequence, was determined and the sequence's probability of alternation, $P(AL)=\frac{r-1}{19}$, was computed. Likewise, for each table, k, the number of color changes was counted and $P(AL)=\frac{k}{180}$ computed. Figure 2 presents the distributions of all the generated sequences and tables as a function of P(AL), along with the distribution of the 150 tables generated by a "nonhuman" random mechanism.

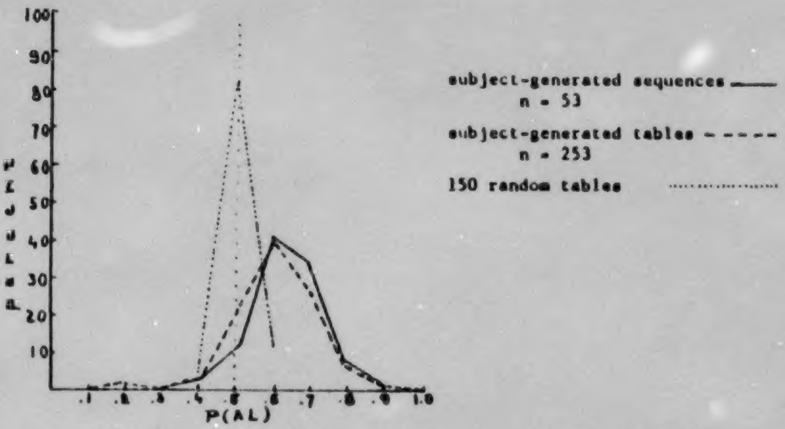

Figure 2. Generation: Distribution of sequences and tables according to their probability of alternation.

Note the similarity between the perception and generation functions. The disparity between the distribution of the random-numbers-generated tables and the human-generated ones is again in the same direction: Humans are over-alternating relative to chance, and the modal value is again P(AL)=0.6.

Random binary sequences of 20 symbols of each category should be distributed approximately normally as a function of P(AL). The parameters of this sampling distribution are: $E[P(AL)]$ 0.51, $\sigma[P(AL)]=0.080$. The mean probability of alternation generated by our 53 subjects was $\overline{P(AL)}=0.61$. This is significantly deviating from chance (for 0.0000001).

Random binary tables of 10x10 with 50 elements of each kind should also be distributed normally as a function of P(AL), with parameters: $E[P(AL)] = 0.51$, $\sigma[P(AL)] = 0.037$. The mean table generated by our 253 subjects, $\overline{P(AL)} = 0.63$, is statistically significant for virtually any level.

DISCUSSION AND CONCLUSIONS

Quite often, when situations are completely random, and especially when alternations are only slightly below chance, people reject the chance hypothesis. Consequently, we look for alternative explanations of the occurrences, and hence, we often overinterpret the world around us and sometimes construct idle theories. This way we commit "type I error", namely, we see pattern were it does not exist, and we impose too much order and lawfulness on the occurrences around us. Psychologically, that kind of error seems natural. One entertains the illusion of coping better with an environment that seems organized.

However, this study indicated that type I error, although quite prevalent, is not the only fallacy characterizing the perception of randomness. The gambler's fallacy is a manifestation of a "type II error" in intuitive judgment. Sequences and tables with a nonchance increase of alternations were generally agreed on as being random. The perceptual fallacy lies in overlooking a significant deviation from randomness. The roots of this fallacy are not hard to understand. One thinks of the law of large numbers, but expects to find the appropriate relative frequencies also in small samples (Tversky and Kahneman, 1971).

The similarity of the performence functions for one and two dimensional stimuli and for perception and generation responses, suggests a general stable image of randomness. The judgment of randomness was insensitive to sample size. The modal value of P(AL) that recurred constantly as an expression of randomness was 0.6. The independence of that judgment on sample size violates the normative statistical prescription, since a probability of alternation of 0.6 is more probable in a random ordered set of small size than in one of a large size. A similar finding of insensitivity to sample size in a somewhat different judgment task is reported by Tversky and Kahneman (1974).

One may regard the perceptual judgment required in this study as a problem of intuitive hypothesis testing. One is confronted with a sample and should draw conclusions about the generating process (the hypothetical infinite population). Subjects' "subjective level of significance" turned out to be inconsistent because of the insensitivity to sample size. Furthermore, symmetrically probable overalternations and overclusterings were judged differently. Acceptance or rejection of a set of stimuli as random, thus depended on the direction of the deviation from randomness rather than on the probability of getting such a sample by chance. This finding has important implications for the teaching of statistics: One should emphasize the need to adhere to an apriori objectively determined level of significance as a standard procedure in scientific research.

Another lesson to be learned from the generation experiments is: whenever you have to produce randomness, use a mechanistic procedure. Never let a human agent, even the most conscientious and well meaning, generate randomness according to their subjective image.

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C

SOLUTION DE PROBLEMES ET MEMOIRE
ETAPES ET CATEGORIES DE LA PENSEE MATHEMATIQUE
LOGIQUE ET REPRESENTATION ; PROBLEMES DE METHODES.

PROBLEM SOLUTION AND MEMORY
STAGES AND CATEGORIES OF MATHEMATICAL THOUGHT
LOGIC AND REPRESENTATION ; METHODOLOGICAL PROBLEMS.

LES CONTRAINTES DE FONCTIONNEMENT DES SYSTEMES MNESIQUES DANS LA RESOLUTION DE PROBLEMES

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Problem solving is considered as a task in which three types of activities are competing : conservation in working memory of informations useful to the problem (data, intermediate results), retrieval from long term memory of notions, rules considered as relevant to the problem, complex cognitive processing such as understanding the problem, planning, inferring, computing... All these activities involving attention, attention has to be divided between them and especially the last one may have detrimental effects on the first ones. In a mental calculus task where at each step preceeding results have to be used in computing the next result, we have shown that pretty many errors appear when retrieving preceeding results beyond the last one computed. Retrieval of notions from long term memory is discussed in terms of distance between the cues available for retrieval in the context of the problem and the cues present in the context in which notions were learned and used previously. The distinction is stressed between availability of a notion and accessibility of a notion, a distinction which as been made in the study of episodic memory.

La résolution de problèmes est considérée de plus en plus en didactique des mathématiques comme une activité ayant sa finalité propre et non plus seulement comme un moyen d'évaluation des connaissances ou une préparation à la construction des notions. Cette orientation amène à s'intéresser plus spécifiquement aux procédures de résolution, orientation à laquelle les travaux de Newell et Simon (1972) ont donné une impulsion déterminante Il ne suffit pas de décrire celles qui sont élaborées par les élèves, il faut simultanément s'interroger sur leurs déterminants, sur les processus qui les engendrent.

Une première classe de déterminants concerne les notions mathématiques connues de l'élève. plus précisément le niveau où il se trouve dans l'acquisition de ces notions. Ce point de vue est essentiel et se trouve au premier plan dans les préoccupations des chercheurs.

Il ne faut cependant pas négliger une autre classe de déterminants, ceux qui tiennent aux contraintes du fonctionnement cognitif, notamment celles qui touchent aux systèmes mnésiques. La résolution de problèmes en effet met en jeu simultanément des activités de traitement (élaboration de représentations, de plans d'action, production d'inférences, de calculs etc.) et des activités de récupération d'informations en mémoire ; récupération en mémoire de travail d'informations concernant le problème, de résultats déjà calculés, recherche en mémoire permanente de connaissances, d'algorithmes ou plus généralement de procédures utilisables.

Nous proposons d'analyser la résolution de problèmes comme une activité d'attention partagée entre tâches et de s'inspirer des résultats des recherches sur l'attention pour formuler des hypothèses susceptibles de rendre compte des différences importantes de performance qu'on observe entre des tâches mettant en jeu apparemment les mêmes compétences, et d'expliquer pourquoi des connaissances sont utilisées dans certains problèmes et pas dans d'autres où elles sont pourtant également pertinentes.

Nous nous limitons ici aux effets que peuvent avoir, dans la résolution de problèmes, les contraintes de fonctionnement des systèmes mnésiques.

LA RESOLUTION DE PROBLEMES COMME ACTIVITE D'ATTENTION PARTAGEE ENTRE TACHES.

Dans la recherche de la solution d'un problème trois types d'activités interviennent concurremment :

1. Stockage en mémoire de travail d'informations nécessaires à la résolution du problème, qui sont conservees momentanément, le temps nécessaire à accomplir la tâche, mais seront oubliees ensuite. Ce sont en particulier les données du problème, des résultats déjà obtenus dont on aura encore besoin par la suite. Ce stockage fait appel à la mémoire immédiate mais dans la mesure où les exigences du stockage dépassent les limites de celle-ci, il met en jeu des activités visant, comme la répétition mentale, à conserver cette information.

2. Récupération en mémoire à long terme de connaissances, d'algorithmes, de règles d'action ou de souvenirs concernant des solutions antérieures de problèmes analogues.

3. Des activites congitives complexes spécifiques au problème à résoudre : construction d'une représentation du problème permettant d'élaborer un plan pour la recherche de la solution, élaboration de plans, déductions ou calculs (et plus généralement opérations de transformations des données), contrôle de l'exécution du plan d'action, etc.

Ces trois types d'activités sont présents simultanément : même si la charge en mémoire peut être diminuée par une représentation externe de certaines informations (par exemple en les notant sur une feuille) il reste qu'une part très importante de l'activité mnésique se fait sans rapport externe. Les recherches sur l'attention (Richard, 1980) ont montré que dans le fonctionnement cognitif il y avait compétition entre ces trois types d'activités, en ce sens que l'exercice simultané de plusieurs de ces activités se traduit en général par une diminution de l'efficacité de chacune d'elles par rapport à la situation où elle s'exerce isolément. Ainsi une information stockée en mémoire de travail a plus de chance de disparaître lorsqu'intervient un traitement cognitif complexe qui empêche sa révision mentale.

On est encore loin de savoir exactement quelles sont les activités cognitives qui ne peuvent s'exercer qu'au détriment l'une de l'autre mais on dispose actuellement de suffisamment de données pour en faire une hypothèse de travail raisonnable.

LES LIMITATIONS DE LA MEMOIRE IMMEDIATE

Le nombre d'éléments que l'on peut maintenir en mémoire à la suite d'une seule présentation est de l'ordre de sept chez l'adulte (Miller, 1956). Ce résultat est connu depuis longtemps, il vaut dans le cas d'éléments sans relation les uns avec les autres (des lettres, les chiffres d'un numéro de téléphone etc.) pour lesquels on demande une restitution immédiate sans intervention d'une activité interférente. Il suffit toutefois d'une activité simple et relativement brève (rechercher une lettre dans une suite) qui détourne le sujet de la révision mentale de la liste à rappeler pour perturber de façon très importante la restitution de celle-ci (Wimbey et Leiblum, 1967).

Par ailleurs Baddeley et Hitch (1974) ont montré que lorsque le sujet doit maintenir en mémoire immédiate un nombre d'éléments supérieur à trois, le temps nécessaire pour effectuer une tâche de jugement se trouvait augmenté et cela d'autant plus que la tâche de jugement était plus complexe.

Les limitations de la mémoire immédiate peuvent se traduire de diverses manières :
- dans la compréhension de l'énoncé : la lecture de celui-ci met en jeu une double activité : une activité de déchiffrage du texte et une activité de stockage en mémoire de travail des éléments pertinents. La première est loin d'être automatisée chez l'enfant et comme toute activité cognitive non automatisée elle peut entrer en compétition avec l'activité de stockage en mémoire.

- dans les traitements effectués : oubli de certaines données non parce que l'élève n'aurait pas été assez attentif dans sa lecture mais parce qu'il n'a pas été assez sélectif dans sa mémorisation des informations, oubli de résultats antérieurs, omission de l'examen d'éventualités alors qu'il a toutes les connaissances requises pour les envisager.

Nous avons choisi une situation de calcul mental pour étudier quel effet a sur l'information stockée en mémoire de travail l'exercice d'une activité cognitive non automatisée.

Le sujet doit faire une suite de calculs additifs figurés chacun par une équation à une inconnue. Dans la première équation la valeur de l'inconnue est calculée à partir de données présentes dans l'énoncé. Dans les suivantes figurent en plus de l'inconnue à calculer d'autres inconnues dont les valeurs ont déjà été calculées : pour effectuer le calcul le sujet doit rechercher ces valeurs en mémoire de travail. On a fait varier systématiquement le nombre d'informations à récupérer en mémoire en vue du calcul ainsi que le nombre d'informations stockées en mémoire mais déjà utilisées : ces dernières, étant devenues inutiles, peuvent être oubliées sans dommage.

L'étude a été faite avec des enfants de 10-11 ans. Si le dernier résultat calculé est retrouvé sans erreur l'avant dernier résultat, et a fortiori les précédents, sont oubliés dans une proportion notable de cas, d'autant plus importante que le sujet a fait plus de calculs entre le moment où le résultat a été calculé et celui où il doit être retrouvé.

Par ailleurs les résultats devenus inutiles perturbent la conservation en mémoire des résultats qui devront être utilisés ultérieurement. A l'âge étudié il ne semble pas y avoir d'oubli sélectif des résultats devenus inutiles.

LA RECUPERATION EN MEMOIRE A LONG TERME

Pour aborder l'utilisation des connaissances dans la résolution de problèmes on peut l'analyser comme un processus de récupération en mémoire à long terme et s'inspirer des recherches faites sur ce sujet dans le domaine de la mémoire, même si elles portent à peu près exclusivement sur la mémorisation d'éléments isolés et non organisés, lesquels sont en cela très différents des connaissances. Dans la psychologie de la mémoire on fait une distinction entre la présence d'une information en mémoire et la possibilité d'avoir accès à cette information dans une situation donnée : une information présente en mémoire, donc non oubliée, peut n'être pas accessible dans des circonstances données et donc être inopérante.

La possibilité de récupérer une information en mémoire à long terme dépend dans une très large mesure de la présence d'indices qui lui sont associés. Cela est encore plus vrai pour des enfants que pour des adultes (Bushke, 1974). L'efficacité de la récupération dépend pour l'essentiel de la parenté entre le contexte dans lequel l'information était présentée au moment du stockage en mémoire et du contexte présent au moment de la récupération (Tulving et Thomson, 1973). Dans cette perspective les connaissances adéquates à la solution du problème peuvent n'être pas évoquées même si le sujet les possède ; par contre peut être évoquée une connaissance ou une règle inadéquates, dont le contexte d'apprentissage est plus proche du contexte constitué par le problème. Le contenu de celui-ci, la forme sous laquelle il est posé, engendrent des indices plus ou moins aptes à susciter les connaissances ou règles d'action pertinentes. Cela explique les grandes différences constatées dans la réussite de problèmes formellement très voisins.

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SPATIAL VISUALIZATION AND PROPORTIONAL REASONING OF EARLY ADOLESCENTS

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Cette étude examine la relation existant entre la visualization spatiale, déterminée à l'aide d'une tâche utilisant du papier plié, et la raissonement mathematique implique dans l'utilisation de proportions. Les résultats à ces deux tâches sont obtenus à partir d'échelles de Guttman, qui reflètent la structure du raisonnement demandé dans chaque cas. La visualisation spatiale a de fortes corrélations avec le raisonnement proportionnel. Parmi les corrélats possibles d'origine cognitive et affective que nous avons étudiés, seules la capacité de traitemement et la restructuration cognitive ont une grande puissance predictive. Une fois ôté l'effet de ces deux corrélats, il reste un effet modéré de la visualisation spatiale sur le raisonnement proportionnel.

A relationship between spatial visualization and complex mathematical reasoning has been recognized for some time (Smith, 1964). However, the nature of this relationship is still not well understood. It could be a direct one or it could be due to elements they have in common, such as general problem solving ability (fluid intelligence, Gf) or processing capacity. The purpose of the present paper is to examine the relationship between spatial reasoning and proportional reasoning, an example of complex mathematical reasoning.

METHOD

The proporti[illegible] reasoning task employed in the current study was the lemonade task (Karplus, Pulos, & Stage, 1980). The proportional reasoning score derived from the eight lemonade puzzles (Karplus, Pulos, & Stage, in preparation) was based on the complexity of a subject's reasoning and formed an acceptable Guttman scale, with coefficient of reproducibility 0.97 and coefficient of scalability 0.89. It ranged from 0 to 3 (Table 1).

Table 1. Proportional Reasoning Score

Criterion	Score
Proportional reasoning never used	0
Only equal integral ratios compared successfully	1
Integral ratios compared successfully with non-integral ratios	2
Non-integral ratios compared successfully	3

The measure of spatial visualization was an individual version of a paper folding task suggested by Thurstone (1951), commonly used in a multiple choice format (Educational Testing Service, 1962). In our specially-developed version, subjects were shown a square piece of paper that was folded once or twice while they were watching. For the next step, a dot was placed on the folded paper and subjects were asked to imagine a hole punched through all paper layers at that point. While still viewing the folded paper with the dot, the subjects were asked to make a drawing that would show the hole locations in the unfolded paper. More details of the task administration and all fourteen items will be published elsewhere.

This report makes use of the eight items for which a correct response required the proper location of three or four holes. The nature of the folds is briefly characterized in Table 2. Each fold was horizontal (H) or diagonal (D) relative to the edges of the square paper, the second fold was parallel (||) or perpendicular(⊥) to the first, the folded section covered the lower section completely or partially, and a partial first fold may have been hidden by a complete second fold. Students' answers were scored correct if all holes were recorded in approximately correct locations and no additional holes were indicated.

Table 2. Paper Folding Item Features

Item Number	Complete Folds	Partial Folds	Hole Location	Layers
5a	H,H(⊥)	--	center	4
5b	H,H(⊥)	--	edge	4
6	D	D(||)	center	4
7b	H	D(hidden)	center	3
8	D	D(||)	center	3
9	D(||)	D(hidden)	center	3
10	D,D(⊥)	--	center	4
11	--	H,H(⊥)	center	4

To examine the possible common correlates of proportional reasoning and spatial visualization, seven cognitive tasks were administered to groups: 1) the Figural Intersection Task, FIT (Pascual-Leone & Burtis, 1974) -- process capacity; 2) the FASP embedded figures test (Pulos & Linn, 1979) -- cognitive restructuring; 3) series completion -- fluid intelligence; 4) the Water Level Task, WLT (Pascual-Leone, 1974) -- field dependency; 5) conservation of volume -- formal reasoning; 6) a vocabulary test -- crystallized intelligence;

7) alternative uses test (Wallach & Kogan, 1965) -- divergent thinking. Tests 2 and 4 measure static aspects of spatial ability different from spatial visualization (Thurstone, 1951; Richmond, 1980), which refers to a cognitive process that deals with relative movements of parts of a figure. A mathematics attitude survey (MAS) was also administered (Stage, Karplus, & Pulos, 1980).

The subjects in the study were sixth and eighth graders in a suburban school in northern California. A total of 125 students was given the paper folding and lemonade tasks, and 78 of these participated in all the group tests. Approximately equal numbers of subjects of each grade and gender were included.

RESULTS AND DISCUSSION

The percentages of correct answers on the paper folding items are presented in Table 3. No significant relationship between sex or grade and these results was observed, so separate data for the various subject groups are not reported.

Table 3. Percentages of Correct Answers on Paper Folding Items (N=125)

Item	5a	5b	6	7b	8	9	10	11
% correct	76	91	51	26	49	37	30	25

It is clear that certain pairs of items had very similar success rates. Easiest were Items 5a and 5b, both of which had two complete perpendicular folds aligned with the square paper's edges. About half the subjects solved the more difficult Items 6 and 8, both of which had a complete diagonal fold and a partial diagonal fold parallel to it. Still more difficult, with success by about one-third of the subjects, were Item 9, with a hidden fold parallel to a complete diagonal fold, and Item 10, with perpendicular diagonal folds. Most difficult of all, solved successfully by only one-fourth of the subjects, were Item Item 7b, with a hidden fold oblique to a complete fold, and Item 11, with two mutually perpendicular partial folds.

For a total score of spatial visualization on the eight paper folding items, each student was given one point for success on one or both items in each of the four pairs. These scores could range from 0 to 4, but actually ranged from 1 to 4, since no subject failed all eight items. The four-level scoring system formed an acceptable Guttman scale with coefficient of reproducibility = 0.93, coefficient of scalability = 0.78.

Spatial visualization and proportional reasoning performances, both measured by scores that form Guttman scales, are compared in Table 4. It can be seen that each score is distributed fairly uniformly over its range, with proportional reasoning somewhat more in the lower part of its range and spatial visualization somewhat more in the upper part of its range.

Table 4. Contingency Table Between Spatial Visualization and Proportional Reasoning (percent, N=125)

Spatial Visualization	Proportional Reasoning 0	1	2	3	Total
1	14.4	3.2	1.6	1.6	20.8
2	11.2	7.2	8.0	1.6	28.0
3	5.6	5.6	6.4	4.8	22.4
4	1.6	7.2	7.2	12.8	28.8
Total	32.8	23.2	23.2	20.8	100.0

The chi-square (39.80 for this table) and the contingency coefficient (chi / $[\text{chi square} + N]^{\frac{1}{2}}$ = 0.49) show a highly significant relationship -- beyond the 0.001 level -- between the two forms of reasoning. At the same time, the relationship appears to be symmetrical in that the asymmetrical lambdas are approximately the same whether proportional reasoning is predicted from spatial visualization (lambda = 0.18) or spatial visualization is predicted from proportional reasoning (lambda = 0.19). This lack of asymmetry suggests that neither of the two variables mediated the other.

To investigate the relationship further, a stepwise multiple regression analysis was carried out for the proportional reasoning score, using the eight correlates (entered in a group) and spatial visualization (entered last) as independent variables. To give an overview of the interdependence of these variables, we first present in Table 5 the bivariate correlation coefficients after eliminating Uses, Vocabulary, and Mathematics Attitude Survey, whose correlation coefficients with proportional reasoning or spatial visualization did not exceed 0.25. It can be seen that proportional reasoning and spatial visualization correlated with one another, but that proportional reasoning had generally higher correlations with the other variables than did spatial visualization. Though the FIT, WLT, and FASP all involve reasoning with shapes, only the FASP score correlated more highly with spatial visualization than did series completion or volume conservation.

Table 5. Bivariate Correlation Coefficients for Seven Variables (N=78)

	Proportional Reasoning	Spatial Vis.	FIT	FASP	Series	WLT
Spatial Vis.	.50					
FIT	.56	.36				
FASP	.49	.41	.31			
Series	.44	.29	.47	.44		
WLT	.41	.27	.52	.29	.27	
Volume	.25	.36	.29	.45	.39	.20

The stepwise regression (Table 6) showed that only processing capacity (FIT) and cognitive restructuring (FASP) increased the prediction of proportional reasoning significantly. Yet spatial visualization, entered last in the regression, also increased the prediction to a substantial and statistically sigificant extent.

Table 6. Stepwise Regression for Proportional Reasoning (N=78)

Variable[1]	R^2	R^2 change	F to enter	P
FIT	.31	.31	25.25	<.001
FASP	.42	.11	10.62	.002
Series	.43	.01	1.06	ns
WLT	.44	.01	.82	ns
Volume	.44	.00	.32	ns
Vocabulary	.45	.00	.21	ns
Uses	.45	.00	.24	ns
MAS	.45	.00	.02	ns
Spatial Vis.	.53	.08	8.54	.005

[1] listed in order of entry

The correlation coefficients in Table 5 already suggested an outcome in which the proportional reasoning, spatial visualization, FIT, and FASP scores were strongly interdependent. The last three together account for about half the variance in proportional reasoning. If FIT, spatial visualization, and FASP are entered in the multiple regressions in that order, their contributions to the variance of proportional reasoning become .31, .14, and .04, respectively.

CONCLUSIONS

The results of this study imply that there is a relationship between spatial visualization and proportional reasoning not attributable to a common relationship to other variables in the quite comprehensive set that was taken into account. Furthermore, neither of these two variables appears to mediate the other. Thus, the two forms of reasoning appear to facilitate one another by a mechanism, if any, that remains to be clarified.

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WHAT KIND OF ORGANIZATION MAKES STRUCTURAL KNOWLEDGE EASILY PROCESSIBLE?

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La construction des structures cognitives demande l'action, c'est le point commun des diverses théories sur l'acquisition des connaissances. Ce principe - autant que les méthodes de l'enseignement des mathématiques qui en dérivent - n'inclut pas a priori, que les connaissances qui s'apprennent de cette manière, soient utiles pour produire des processus cognitifs; cette à dire qu'elles seront capables de créer des activités pour appliquer les connaissances et à contribuer pour apprendre le savoir ultérieur. Un point essentiel de cela est, qu'on a besoin d'une suite d'évènements bien dirigés, qui se mettent en scène par des actions consécutives d'un "agent". Nous allons discuter la question, quels principes d'organisation des connaissances pourraient favoriser la construction des structures cognitives orientées vers l'action.

1. STRUCTURE AND PROCESS

All human perception depends on certain grounds of shape, space and time, which are the very first formal principles of all phenomena, as was pointed out by Kant (Kant, 1975). Space and time are the schemata and conditions of cognition, not originating from but presupposed by our senses. Nevertheless, they are subject to the logical inferences of the intellect which takes account of space when considering objects and of time when considering states. We use space to express (statically) the interrelations between things, thus yielding *structure*, and we use time to express (dynamically) states and alteration, by this yielding *process*; and all kinds of mathematical considerations can be expressed in terms of structure and process. (What is called a category, for instance, is given by a set of objects together with a set of morphisms.)

Again, we find these principles of structure and process in various models of knowledge representation: To express interdependancies between contents of knowledge which in some respect refer to the same certain issue, we use the notion of *schema* which can be considered as to arrange certain sets of concepts within the "mental space" of all concepts already formed. So, when speaking of "knowledge" in the following, we shall always refer to *conceptual* structures, as *schemata* in the sense of (Skemp, 1979) and, in a respect somewhat different (see below), as *memory schemata* in the sense of (Bobrow/Norman, 1975), and to what is called the *information processing* level or likewise symbol processing level, where meanings become attached to the mental entities we process: It is the symbolic representation which enables man to deal with a manifold of phenomena without having to reiterate them in their totality, and it is the schema which is assumed, by many authors, to be the primary organizing unit of meaning and processing of information.

But, according to our view of structural and process-like aspects herein, we find two different types of "schemata" to be distinguished: a "relational" and an "operational" type; the first one, by use of descriptions, stating relations between the concepts involved, and thus giving rise to *understanding*. The second "active" type evolves orientation from descriptions ("structural knowledge") to process; e.g. in assimilating realities or conceptual interdependancies *to bring about understanding*, or in reorganizing structural knowledge to *integrate* new situations and experiences (accommodation), or to *produce action* by directory of the disposable knowledge: It needs process to make structural knowledge effective.

This second type of a schema would, for example, be called "schème" by Piaget (vs. "schéma" in case of type-1) or "operative schema" (vs. "figurative schema") by Inhelder; Bartlett would rather speak of "active, developing pattern" or "organized settings", Furth would speak of "operative plans", Lindsay, Norman, and Rumelhart of "action schema", and Neisser (who refers to the first type as "stored plans of actions" in the sense of Miller, Galanter, and Pribram) speaks of "stored plans *for* action which direct their execution". (All references see Kluwe, 1979, pp.20-23; see also Skemp, 1979, p.219.) Bobrow and Norman conceive schemata to be "active processing elements" which can become active if requested (Bobrow/Norman 1975, p.132); and what in (Skemp, 1979) is called a "director system" seems to be related to this second type of a schema.

All schema models of the first type commonly serve as structural representation units which organize knowledge and are *subject to accommodation* (or restructuring), while schemata of the second type are involved in process: they are models for "instances" *which* e.g. *accommodate* represented knowledge, and *which direct action*.

2. PROCESS *vs.* ACTION

A most important difference between considering mathematical processes, as far as mathematics itself is concerned on the one hand, and from the mathematician's or student's viewpoint on the other, lies in the fact that what is process in the first case appears as *action* of a person (an "agent") *evolving* the process, by a sequence of acts. And while,in difference to structure, process is coupled with a *direction* (of proceeding), is action coupled with *intention* (whatfor to proceed); while mathematical process relates to logical rules, does action submit to psycho-logical influences: The agent has to make decisions what intermediate goals are to run up to, and what could be "means" to reach "ends". And as a basis, and motive, for his decisions he uses what we have been calling type-1 resp. type-2 schemata.

But what is the origin of such knowledge structures, and how do they develop? Some of the theories dealing with the acquisition of knowledge agree in the point that to build up cognitive structures action is required ("to comprehend is to operate"), including mental action by symbolic operation, and basing on the mechanisms of what Piaget has called empirical and reflective abstraction (e.g. Piaget, 1975, pp.87-89). However, this principle, as well as derived methods for mathematics education, would it include the fact that the structural knowledge acquired from this is process-oriented, i.e. is able to produce actions which yield application and contribute to further acquisition of knowledge? Does comprehension already provide adequate action schemata?

Resulting from reflective abstraction, a conceptual schema is not of a special style as a sensori-motor schema normally is (e.g. typewriting schemata would hardly serve as, or be extendable to, practicable schemata for playing piano). But instead, one of its most important features is it to be *general*, and, in fact, very often it is the degree of generality of a conceptual schema that becomes extended in the process of learning mathematics. Action schemata must be general to apply to a broad range of situations, which is a requirement of *economy*, and they must be general to be applicable to situations not having appeared during the process of their abstraction, which is a basis for *transfer*. So, what can be done to pursue these principles of generality in comprehension during processes of learning mathematics?

The following two examples are chosen to elucidate the role of general action schemata not being focussed to special context, as well as of psychological concerns influencing decisions when knowledge is put into process (as an agent proceeds).

3. TWO EXAMPLES

1) Having learnt some relation between the side lengths of a right triangle, say, $a^2 + b^2 = c^2$, certainly a student will soon be able to determine b, if for example a and c are given, in generating process by use of an action schema like

first: isolate the unknown
then: insert known information

which, in such a general shape, usually was acquired earlier and perhaps has to be extended to the new context of quadratic equations.

2) But what about someone having recently learnt how to integrate certain classes of real functions, including the rule for partial integrating, and is given the task

$$\int cos^2x dx = ?$$

Analyzing this task, which soon turns out to be a problem, could give us some insight, so let us see. We'd try "to integrate" (wouldn't you?) in the *first approach:*

$$\begin{aligned} \int cos^2x dx &= \int \overset{f'}{cosx}\ \overset{g}{cosx}\ dx \\ &= [\overset{f}{sinx}\ \overset{g}{cosx}] + \int sinx\ sinx\ dx \\ &= [sinx\ cosx] + [-cosx\ sinx] + \int cos^2x dx \end{aligned}$$

which carries back solution of the task to the solution of the task. Problem! Remembering (from some part of structural knowledge) that $sin^2x = 1 - cos^2x$, we would probably come to the *second approach:*

$$\begin{aligned} \ldots &= [sinx\ cosx] + \int (1 - cos^2x) dx \\ &= [sinx\ cosx] + \int 1 dx - \int cos^2x dx \end{aligned}$$

which again seems to yield a problem of self-reference. So, if we still follow the command (INTEGRATE!) of the symbol "$\int$" we are getting into a deep-end. What to do? - Now let us see how general our action schemata are! Do we remember that for example $\sqrt{t}$ could signalize: TRY TO RADICATE! or could indicate: TAKE IT AS AN OBJECT!; i.e. that the symbol "$\sqrt{\ }$" is of a twofold psychological nature, showing *command for proceeding*, as well as *declaring an object* (a number)? Then we would possibly be able to transfer this knowledge (which is processible in more than one respect) to the recently introduced symbol "$\int$", and switch from transformation of terms to equivalent-transformation in the *third approach:*

$$\int cos^2x dx = [sinx\ cosx + x] - \int cos^2x dx$$

$$\Leftrightarrow \quad 2\int cos^2x dx = [sinx\ cosx + x]$$

$$\Leftrightarrow \quad \int cos^2x dx = \tfrac{1}{2}[sinx\ cosx + x]$$

and there we are!

4. ACTION-FAVORING ORGANIZATION OF KNOWLEDGE

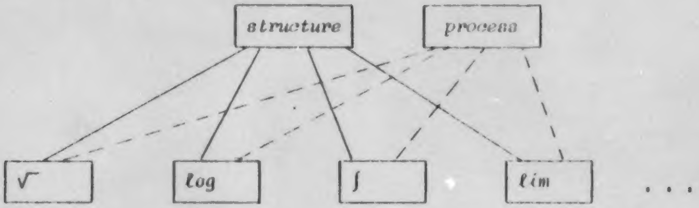
What does it mean to induce process from structural knowledge? At first, it could mean that, determined from a given task, a certain "goal state" is given to be reached, starting from some "present state", in a sequence of intermediate states, a "path" (Skemp, 1979, p.168). We could likewise say that a sequentialized, directed grouping of events, carrying on from state to state, is needed; in this making up the process which an agent has to evolve (see above), in a certain sequence of acts.

Not in every case, however, is a goal state actually known from the task formation: If the task is, say, to prove a certain assertion (and supposed it is valid), one has to find a sequence of inferences, the application of which being the events figuring the process. But, for (counter-)example, in the task above of integrating cos^2x, the goal state has to be found *during the process* (which is not merely a straight-forward computation determined by an algorithm as e.g. for x^2). So in this case, one has to use "means", not knowing to what actual ends! To evolve process, here, does *not* mean to an agent: to compare a goal state with some present state, make a plan, and figure out action, but "merely": to do *something* to perform the task. Hence, to progress to an unknown goal state, it needs various knowledge how to put knowledge into process at all, *how to move from* a position (present state) rather than how to reach a final order (goal state).

So what kind of organization of an agent's knowledge could make it easily processible? As a *first organizational principle*, we suggest to establish *twin* representations of mathematical concepts in a schema, using descriptions showing process as well as structure (e.g. "$\sqrt{2}$ stands for a process: to seak for a non-negative number which, being multiplied by itself, gives 2" *and* "$\sqrt{2}$ declares an object: *it is* the non-negative number which, being multiplied by itself, gives 2" etc.). In paragraph 3. we saw that within the same task, "$\int$" calls process where cos^2x is the operand, as well as, a few steps later, it declares an object to be operated on as a whole, and which could be envisioned to be the result of the above process. Very similar situations would in fact appear upon $\sqrt{\ }$, $\cap$, *log*, *lim*, μ (from μ-recursion), etc., and lead to similar decisions, though occuring in quite different mathematical contexts.

As a *second organizational principle* we thus suggest to arrange *laterally* such special-context schemata, which are consecutively acquired in the course of mathematical instruction; i.e. to evolve *links* between relational descriptions being of similar shape, independent of context. (In the cases above, it is the context, not the functional aspects of symbols, which really differs.) Such links can be revealed by comparing discussion, and by reflective mental activity, yielding insight into general action structures and thus give rise to *general descriptions* for action schemata as postulated in the second paragraph. And this principle can immideately be used in further acquisition of knowledge. (When teaching mathematics at school, in following this line, I once used as a key sentence for the formation of some lessons: "In what respect is *log* for *exp* the same as is $\sqrt{\ }$ for the power?")

In the whole proceeding, keeping aware of structure and of process should yield links to a higher-order view of the potentials involved, leading to a schema as sketched below, and allowing economical representation of knowledge, and transfer, by identifying analogies and similarities, which is a process of *abstraction* (from special-context schemata), and of *re-concretization* (in a particular special-context schema).



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PROCESSES INVOLVED IN THE SOLUTION OF NON-ROUTINE PROBLEMS

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The talk will focus on the main findings of a three year research project into the problem-solving behaviour of schoolchildren (age : 13 - 16) attempting to solve non-routine mathematical problems.

We will characterize some general features of the observal or infered problem-solving behaviour using a particular framework for analyzing thinking out loud'protocols. These will be compared with results from other research works using different modes of protocol analysis.

Finally, methodological problems encountered using the interviewing technique will be discussed, as well as, some of the attempts at overcoming these difficulties.

EVALUATING PROCESS ASPECTS OF A MATHEMATICAL CURRICULUM

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Cet article présente un travail sur l'evaluation d'un nouveau cours, qui appuyé sur les processus mathematiques de généralisation, déduction et représentation. Sous ces titres, on a catégorisé les devoirs d'élèves contenus dans notre cours. D'une même façon, on a décrit dans cet article le contenu du test, qui demande l'utilisation des mêmes processus dans de nouveaux contextes.

Des groupes d'élèves de la fin de la première et de la deuxième année du cours, dans les écoles du Projet et dans d'autres écoles, ont passé les tests, et on a analysé les resultats a l'aide d'une méthode 'differentielle', pour permettre la comparaison des groupes non-pareils.

Les premiers resultats montrent une supériorité pour les groupes du Projet dans les processus les plus importants du cours, ceux de généraliser et d'expliquer, et aussi dans l'exactitude de raisonnement, qui etait moins appuyé dans le cours. Les questions concernant l'utilisation des nouvelles répresentations ne montrent pas un effet qu'on puisse attribuer au nouveau cours. On va confirmer ces resultats provisoires.

The first author's paper to the 1979 Osnabruck conference of this group reviewed research and development relating to process aspects of mathematics, that is, activities of mathematical investigation, problem solving, proof, representation, generalisation and abstraction (Bell, 1979). The present paper reports work at Nottingham on the development and use of test material designed to measure progress, in this dimension, of pupils aged 11-13 following a course with a particular emphasis on process aspects of mathematics, the South Nottinghamshire Project (SNP) (Bell, Wigley and Rooke 1978-9).

There have been some previous attempts at similar evaluations. Williams and Fogelman (1972) carried out three comparisons between groups, aged 7-8, taught sets and logic with Dienes' Logic Blocks, and corresponding control groups, using certain intelligence tests as the criterion measures. The results were inconclusive; in the longer but less well designed experiment, gains were observed on some of the measures, but in a shorter but more rigorous comparison there were no significant differences. A substantial curriculum experiment in Calgary, in which 20 teachers were trained at a summer school in the use of South Nottinghamshire Project and other materials for certain curriculum topics, the experimental classes showed significantly better progress than the controls both on normal 'content' tests and also on the SNP test of general mathematical strategies (Brindley 1980).

During 1978-9 further test material was piloted at Nottingham (Galbraith 1979, Horton 1979). Galbraith's work aimed mainly at gaining further insights into pupils' understanding and use of proof, and he has subsequently continued this work in Australia (Galbraith 1981); Horton produced four 50 minute written tests for use in curriculum evaluation. It is the results of the trial administration of these tests to pupils in a few Project and non-project schools which are to be reported here. We shall, however, analyse more fully than hitherto the nature of the experimental course and attempt to relate the test outcomes to the curriculum emphases. This was done descriptively for the first year of the SNP course in a previous study (Bell 1976). Here we shall deal with the second year, since a main point of interest in the test results are the gains during the second year.

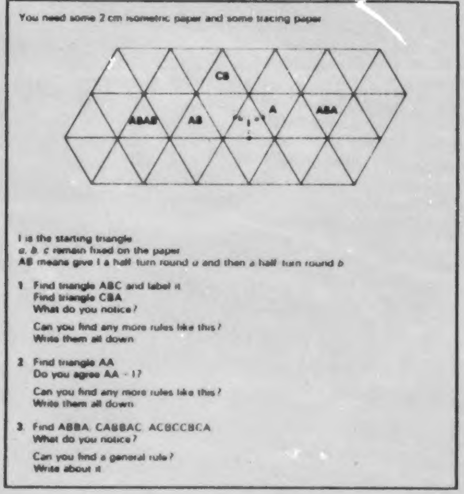
PROCESS ASPECTS OF TASKS IN THE SNP SECOND YEAR COURSE

A rough classification of the 120 pupil tasks in the main part of the SNP second year course, excluding those in the individual Number Skills booklet, shows the following distribution:

- about 1/3 are Generalisation tasks
- about 1/3 are Representation tasks (also involving generalisation)
- the remainder are divided among Classification, Choice of Measure, Optimisation, and Concept and Algorithm learning tasks.

The typical Generalisation task involves the proposing of some rule governed situation for investigation, in which pupils generate examples conforming to the stated rules, seek patterns of relationship among them, formulate and test a generalisation, then seek to explain or deduce it from the given rules, i.e. from the initial description of the situation. An example from the second year course, *Calculating the Mean*, aims to establish the validity of using a so-called 'fictitious mean'. The theorem is Actual mean = Fictitious mean + (mean of the deviations). The task begins by asking for the calculation of the mean of a set of five single digit numbers, 4, 1, 3 ..., then the mean of 14, 11, 13 ..., then 34, 31, 33 ... and so on; the theorem is induced from a set of given examples and the pupil is asked to test it further with examples of his own choice. Then he is asked to *explain why it works*. In this case, though not always, he is then asked to practise its use in a short set of examples. Another generalisation task, Making Up Symmetry, gives a set of relationships to investigate but with less direction towards a particular generalisation. It asks for a listing of the types of symmetry (number of planes of symmetry) possessed by objects made from face-linked cubes. The invitation is to identify what numbers of planes are not possible, and to explain why.

Representation tasks are those in which the relation between a situation and its corresponding symbolism or diagram are established or exploited. The main types are (1) those in which an algebra is established to describe aspects of a geometrical situation, (2) those which relate numerical data, the shape of the Cartesian graph, and the corresponding algebraic formula, and (3) those in which the algebra of line segments and their number-pairs (vectors) is related to the geometry of quadrilaterals. An example of the first type is Triangle Algebra, in which the spaces of a triangular tessellation are labelled according to the sequence of movements needed to transfer a triangle from the starting position to that space, and the identity properties of the resulting 'words' (such as ABCCAB) are related to the geometry of the figure (see below).


CB
ABAB
AB
A
ABA
I
a
b
c

You need some 2 cm isometric paper and some tracing paper

I is the starting triangle
a, b, c remain fixed on the paper
AB means give I a half turn round *a* and then a half turn round *b*

1 Find triangle ABC and label it
Find triangle CBA
What do you notice?

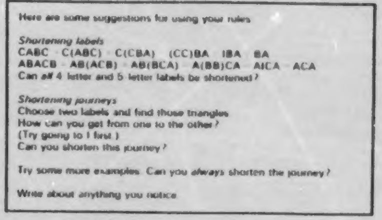
Can you find any more rules like this?
Write them all down

2 Find triangle AA
Do you agree AA = I?

Can you find any more rules like this?
Write them all down

3 Find ABBA, CABBAC, ACBCCBCA
What do you notice?

Can you find a general rule?
Write about it



Here are some suggestions for using your rules

Shortening labels
CABC = C(ABC) = C(CBA) = (CC)BA = IBA = BA
ABACB = AB(ACB) = AB(BCA) = A(BB)CA = AICA = ACA
Can *all* 4 letter and 5 letter labels be shortened?

Shortening journeys
Choose two labels and find those triangles
How can you get from one to the other?
(Try going to I first.)
Can you shorten this journey?

Try some more examples. Can you *always* shorten the journey?

Write about anything you notice

Classification tasks generally request the collection of a complete set of objects of some type; the process requires the establishment of subcategories and some implicit generalisations about what subtypes are possible. Choice of Measure occurs, for example, in the Statistics section, where pupils, in pairs, are asked to estimate, then measure, a set of lines, and to consider who has the 'best set' of estimates; the mean and spread are considered as possible measures for determining the 'best set'. An Optimisation task features in the Circles and Area section; a problem requires the determining, by trial and improvement, of the largest grazing area for a tethered donkey in a triangular field, the choice being of where to fix the central peg. Concept and algorithm learning tasks probably do not need illustration.

EVALUATING STRATEGY LEARNING

We now turn to consider the administration of the process tests, and the results obtained. The aim of the test was to evaluate the strategies of Generalisation, Explanation, Proof, Representation and Classification. (In addition to these, two items were included to test aspects of the interpretation of 'real life' data displayed in a table or graph. These had only a relatively small part in the course, but we wished to increase the emphasis on realistic applications and wanted the test to be usable for future evaluation of this aspect.)

Design of the evaluation

Two of the tests (SA,SB) contained six questions each, and two (LA,LB) comprised three longer questions; each question contained several markable items. (The problem in designing process test items is that the strategies to be evaluated only come into play when a sufficiently complex situation has been generated, so that items of the usual short length are not possible. Comment will be made below about this.) The pairs of tests are eventually intended for use in a crossover design, so that the same year group can be tested annually for at least three years without meeting the same test each year. The present stage is that at school X, one half of the 1978 entry has taken test SA in June 1979 and test SB in June 1980, while the other half has taken SB in 1979 and SA in 1980. School Y has used the longer question tests LA, LB in a similar way. In addition, each test has been taken by groups of roughly comparable first and second year pupils at neighbouring schools. The aim of the present analysis of results could not be - and is not - to establish precise comparisons, but rather to see whether the relation between the process-oriented curriculum and the tests is sufficiently robust to show through the comparison of unmatched groups; and to investigate what methods of test design and analysis might help in the future if, as is likely, imperfect matching of groups is inescapable.

Content of the tests

Tests LA and LB are fully described, with illustrations of responses at different levels, in Horton (1979), and similar information about test SB is contained in Quarry (1980). We shall therefore use test SA for discussion here. The six questions of test SA will be described. Stamps is shown:

> Suppose you have a lot of stamps of value 5p and 7p but no others. You can make up various amounts of postage from these. For example, you can make up 17p as 7p + 5p + 5p.
>
> Jane says 'You can't make up an amount of 13p'.
>
> Explain fully why Jane is right.

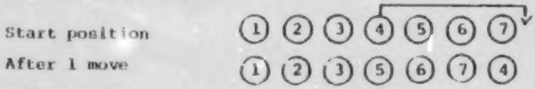
It aims to test an aspect of explanation, in particular the ability to present a complete argument. The marking levels distinguished are (a) a complete, explicit argument, exhausting all possibilities, (b) partial attempts to explain or to illustrate by relevant examples, and (c) no relevant or correct comment. Most pupils who took the test showed signs of understanding the impossibility of obtaining 13p, but this is not possible to assess reliably from the scripts; what can be assessed is the degree of explicitness of the argument, and thus its quality as a communication.

Add and Take tests the ability to recognise a general and explanatory argument as compared with one based on two cases. The question begins, Choose a number; add it to ten; take the original number from ten; add the two results. Two cases are to be produced; then the pupil has to choose the better of these two arguments and explain the reason for his choice:

> JANE says,
> 'Begin with 1, answer is 20.
> Begin with 9, answer is 20.
> So begin with any number between
> 1 and 9, answer will be 20.
> The answer is always 20'.

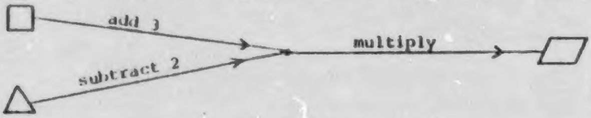
> BRENDA says,
> 'If n is the number you choose,
> you have 10+n, and you add 10-n.
> You add and take the same number
> so you will always be left with
> 2 tens. The answer is always 20'.

Move Along is aimed at testing the ability to recognise and state the move-patterns or rules emerging from a simple counter game. A row of seven counters is given, and the move allowed is to move the middle counter to the right hand end. This is illustrated:



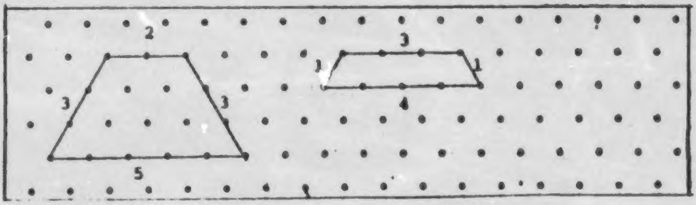
Some rows of blank counters are given for the pupil to experiment with, then he is shown four positions (John's, Mary's ...) and asked to identify errors in them. This is to provide sufficient experience to make possible the focal question: "Can you see any rules which tell you whether a position is possible or not? ... If so, state them as clearly as you can". 'Generalisation' marks are given for the recognition and statement of rules equivalent to 'Numbers less than 4 can't move' and 'Numbers 4 to 7 move cyclically'.

Add 3 is a Representation item. It attempts to test the ability to accept and work with a novel diagrammatic way of representing some number processes. The pupil has to use the given diagram, and also adapt it to represent a new rule.



Adding a Nought asks for an explanation of the familiar principle that a number may be multiplied by ten by 'adding a nought'. Countries presents tabulated data of population, area, income and cost of living for six countries and asks several questions requiring reading, comparing and, finally, choice and use of a measure of being 'better off' from the data available.

Space does not permit description of the remaining tests, but a further item will be shown to illustrate Representation. Roofs offered a coding for roof-shapes drawn on an isometric dot grid, as shown:



Some possible and impossible roofs were then to be drawn from their given codings (2242, 3251 ...), then there was space for chosen trials, and the request to find and explain rules for deciding whether or not a given code will produce a roof.

PREDICTIONS AND RESULTS

It is clear that the strategies being tested in these questions do not all figure equally strongly in the curriculum described above, though they are all aspects of the mathematical process. The completeness of argument tested by Stamps is a strategy which features in several Open Investigations units, but the novel symbolism of Add 3 is not very much like anything in the course; the use of tabulated data, as in Countries, occurs a little, but probably not much more than in a normal curriculum. On the other hand, the making of generalisations and of exaplanations by processes such as those of Move Along and Add and Take do figure strongly in the special curriculum; and Adding a Nought is close to the explanations of decimal place value which are given.

Thus one may predict differential differences between the scores of project and non project groups, according to whether the strategy tested by the particular item is featured in the special curriculum or not. We shall refer to 'featured' and 'non-featured' strategies. There is also a further source of such differentials: within the questions which test a curriculum-featured strategy there are in several cases early parts which require only the correct following of given rules, to generate the examples from which generalisations will be made. These will be called non-strategic items, and they also are predicted not to be affected by the special curriculum. The testing of these predictions is not prevented by the non-matching of the groups to be compared.

The analysis of the data so far completed for test SA, the general pattern of group differences, for non-strategic and non-featured items, is school X, year 2 > school X, year 1 > other schools ($X2 > X1 > OS$), the differences being non-significant. But the featured, strategic items appear to show a different pattern, in which the X2 group is superior to the others at a high level of significance. In particular, on all the tests the Generalisation and Explanation items, and the Representation items of the same type as Roofs, stand out from the general pattern of results as those on which the project groups show differentially superior performance. Superior performance was also shown on an item which asks for identification of which of three given statements was needed for the drawing of a conclusion (a strategy not featured in the curriculum). On the other hand, Add 3 and a similar item requiring adoption of a new symbolism (e.g. $Px(y)$ for $y - x$) did not show superiority for the project group.

The results of the full analysis of these tests will be presented at the Conference, and the tests and analyses themselves will be published later this year from the Shell Centre.

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ON STAGES IN MATHEMATICAL THINKING

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Several authors have presented ideas upon the nature and stages of mathematical thinking.

One question is, whether there is a correspondence between the stages in mathematics (we know for example the concept of thypes) and the stages in mathematical thinking. A second question is, whether there is a correspondence between these stages and stages of learning mathematics. That means, is a theory of stages in mathematical thinking a theory of learning (mathematics).

The presentation will make some remarks upon these questions.

ON MATHEMATICAL THINKING AND UNDERSTANDING

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Plusieurs auteurs (comme Davis et McKnight, 1979; Skemp 1979; Vermandel et Cohors-Fresenborg, 1978) ont développé, ces dernières années, des analyses très intéressantes sur les différentes sortes de compréhension et la nature de la pensée en mathématiques. Nous cherchons à déterminer dans quelle mesure ces conceptions sont utilisable pour interpréter les processus de pensée des élèves dans leur travail habituel sur des objets mathématiques. Après avoir pris pour domaine d'expérimentation dans nos recherches précedentes celui des fractions et de la proportionnalité, nous observons maintenant des élèves dans des activités de preuve de théorèmes de géométrie.

Recently several authors (e.g.: Davis & McKnight, 1979; Skemp, 1979a; Vermandel & Cohors-Fresenborg, 1978) have presented very interesting ideas upon the kinds of understanding and the nature of mathematical thinking. It seems reasonable not only to explain more precisely these (and other) concepts by means of isolated examples taken from several topics of (school)mathematics. But we also try to verify whether the concepts are useful to the elucidation of those thinking processes which we can observe in schoolchildren when they are working with mathematical objects. For this purpose in former experiments we ordered problems taken from the topics of fractions and ratio in series which were related to the idea of 'instrumental - relational - formal understanding', and we looked at whether this concept works (Hasemann, 1980, 1981). In a new experiment we observed students who were proving geometrical theorems, and we used the concepts mentioned above to explain our results. Some aspects will be given in this paper.

One central point in these theories is the concept 'schema', or 'frame', respectively; in our experiment we focussed on the schema 'proving geometrical theorems'. The data were taken from a written test and in interviews. Before taking the written test from a group of students of middle abilities (grade 9) we tried to construct the schema in the students' minds by treatment. In the lessons we followed partly the ideas of Gal'perin (cf. Butkin, 1972), i.e., we paid particular attention to a long period of orientation, and we instructed the students to unfold the premiss completely.

As a starting point we chose a rectangle with its diagonals. Among others, the students formulated the following propositions:

(i) The diagonals in a rectangle divide each other in half.

(ii) The diagonals in a rectangle are equal in their longitudes.

The students recognized the necessity of a proof for these statements, but they didn't know how to do it. Therefore formal proofs were worked out for (i) and (ii). The students learnt that proving is a sequence of actions:

0. Given are a geometrical situation (the premiss), and a proposition.
1. Use symbols.
2. Draw all conclusions from the premiss.
3. Draw further conclusions by using theorems you know.
4. Test whether the proposition follows from these conclusions.
5. If yes, write down the proof in a chain of logical arguments, otherwise look for more conclusions to fill up the gap.

Although this schema looks like a plan of action, it is quite vague. Most of the points are (under)schemas which call for further action, as for example: to remember mathematical facts or theorems; to check whether their premisses are satisfied; to apply these theorems and to formulate the results with the symbols chosen; etc.

In the written test as well as in the interviews the students were asked to do two items:

(1) Given are a parallelogram and its diagonals. Prove that the diagonals divide each other in half.

(2) Given is a square. Prove that the diagonals are at right angle to each other.

The proof of (1) can be given directly by using the plan of action used in proposition (i), while for (2) a re-construction or a generalisation of this plan is required. The results of the written test were as expected: the students were quite good with item (1), but nearly all of them failed with (2). Most of them remembered the plan of action from the lessons and used it for item (1), but they copied this plan for working out (2). Some believed to have solved (2) by this procedure, others searched in vain for additional arguments, or they just gave up.

We shall first interpret these results with the definition of 'mathematical thinking' which was presented by Vermandel & Cohors-Fresenborg (1978). Accordingly, "a mathematical action is every action carried out on an extrapolating schema". This means that such actions should not be called 'mathematical' which are carried out on the base of a plan represented in a schema (as, e.g., the applying of an algorithm, a formula, or a theorem), but by mathematical actions these plans are extended, changed, or generalised. By this definition, in our example mathematical actions are not needed to solve item (1), but for item (2). To develop the schema 'proving geometrical theorems' in the students means to stimulate them to do mathematical actions. In this direction the instructions were evidently not very successful: rather the students just copied their old plan of action. But this result was not surprising to us because the students are not used to carrying out mathematical actions in the sense mentioned above; such actions are demanded very seldom in normal classroom instruction.

According to Skemp's 'Model of Intelligence' (Skemp, 1979a) the proving of theorems is located in the director system Δ_2 (Skemp, 1979b, p. 200). But Skemp is not only interested in mathematical actions; his model encloses all kinds of intelligent goal-directed activity. The distinction between two director systems (Δ_1 and Δ_2) is related to the construction of schemas (Δ_2) and the developing of special plans for action with physical objects from these schemas (Δ_1). Accepting Skemp's model, this means that the students in our experiment failed in Δ_2-activity (but of course it would be absurd to conclude that our students are unable to do Δ_2-activity at all).

Although Skemp describes in principle the ways to build up and to test the schemas (Skemp, 1980, p. 6), it seems to me impossible to derive from his model concrete methods of doing this. A teacher may possibly find out how far a student has understood a special content - whereby to understand a concept is to connect it with an appropriate schema. But this presumes that the teacher has an exact mental image of the schema, not only with regard to the content (the conceptual structure), but above all with regard to the present state of the schema in the student's mind.

Another aspect was explained by Herscovics in his critical comments on Skemp's model (Herscovics, 1980): In fact, when concerned with mathematical problems most students have no sensor to compare their own present state and the goal state; the student "does not as yet possess the means to judge whether or not he has achieved the desired goal state and is dependent on the teacher ... Here we find that the sensor and the comparator are external to the learner" (Herscovics, 1980, p. 3). In our experiment this effect can be shown especially in the interviews: exepting isolated students, most of them were not able to decide autonomously whether their proof was finished or not, although they could write down all relevant mathematical arguments in the correct order. An analysis of the interviews shows that at the end in most cases the interviewer himself summarizes the proof, or asks the student to do this (by which for the student the hint is indirectly given that the proof now is finished).

Theoretical models are useful for explaining the student's behavior. But for practical work it seems more essential to describe the mechanisms which manage the functioning of mathematical thinking. For this purpose Davis & McKnight (1979, p. 95 foll.) list 12 "hypothetical mechanisms that appear to be useful in discussing various aspects of mathematical thinking". Some of these mechanisms are also useful for us. Davis & McKnight differentiate 'sequential processes' and 'Gestalt-processes', whereby the 'frames' belong to the latter. If in a problem-solving-process an appropriate frame is retrieved early, it can guide the problem-solver in looking for that imput data which are most essential; "all such seeking ... is often called 'top down processing': the schema is itself guiding the data collection" (Davis & McKnight, 1979, p. 100).

The 12 'hypothetical mechanisms' are based on error analysis, and another source of basic metaphors is the "sophisticated computer programming". Following Davis & McKnight, the frame 'proving geometrical theorems' might be constructed in the student's minds in this way: First, 'proving' is presented by the teacher as a 'visually-moderated sequence' (VMS); this means that the students learn that to prove is to find by help of their mathematical knowledge and some logical rules a route from a given situation (e.g.: a rectangle with its diagonals) to a proposition (e.g.: prop. (i)). This procedure is a 'VMS' because one need not be able to describe the entire route merely from memory, without the prompts given by key visual inputs along the route; as for instance: unfolding the premiss; taking well-known theorems and checking whether they are applicable; drawing conclusions from these theorems; etc. However, the frame 'proving geometrical theorems' is not necessarily constructed in this manner. Therefore Davis & McKnight hypothize the existence of another mechanism by which a VMS will become a frame. Unfortunately there is no evidence yet how this mechanism works. Possibly this process occurs after the student has solved a novel problem: strong problem-solvers are often lost in thought for long periods after they have solved a new problem (the 'Nowell phenomenon'). Maybe, in this time of past-task contemplation the truly creative process takes place: It "is the ideal time for (the student) to name and describe some of the key parts of his experience ... this is the time for (him) to extend his meta-language vocabulary" (Davis & McKnight, 1979, p. 101).

The 'hypothized mechanisms' are useful for practical work as far as their existence and running may be verified in real thinking processes. In fact, in our experiment we got certain error-categories in the students' proofs (schema-errors, errors in procedures) which might be related to such mechanisms. In addition, the list of mechanisms gives a field to search for possible (error-)strategies. This eases the location and the classification of errors.

In my opinion, the theories mentioned here all allow the description of mathematical thinking processes in more detail. But there remain two general problems which are closely related. One was mentioned also by Skemp (1979a, p. 165): "Though a coherent explanation of diverse post events is satisfying, the survival value of any model lies in its ability to help us direct our actions aright on future occasions. This requires that it can be used to make predictions." In fact, Davis et al. (1979), for example, made very precise predictions based upon their own theory. In our experiment we could of course give general predictions on the students' results, as e.g.: "item (2) is harder than item (1)". But we found it rather difficult to make real and concrete predictions in detail, e.g.: "a student who reproduces in item (1) very precisely the plan of action which was presented in the lessons (i.e., who uses the same symbols, notations, and, above all, the same roundabout ways), is very likely to copy this plan of action also in item (2)". This prediction can be made because 'proving' first was presented as a 'VMS', i.e., some students who did not achieve the general schema (or frame) might believe that to prove means to work out step by step a special plan of action. But this is just one prediction (which can be verified easily); how should we make more which really fit the mechanisms described, and what does 'verify' or 'falsify' mean? That is a methodological problem which should be strongly emphasised.

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THE PROBLEM OF REPRESENTATION OF MATHEMATICAL KNOWLEDGE

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Stimulated by some work with deaf children's learning algorithmic concepts the more general theoretical problem of the role of language in representing mathematical knowledge is posed. There has to be distinguished between social language (the natural languages) as carrier of interpersonal information and possibly very different kind of «metalanguages» as means of internal representations of knowledge. Especially the problems of representation of action-structures and algorithmic concepts is inquired.

LOGIC, AUXILIARY FORMALISM AND GEOMETRY BY TELEPHONE CALL.

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Dans une étude antérieure (1980), nous avons montré que le formalisme auxiliaire décrit par Cohors-Fresenborg (1978) peut favoriser le développement de communications structurées chez des enfants âgés de 7 à 8 ans. L'un de nous a étendu cette observation à l'emploi d'autres formalismes auxiliaires (F.L., 1980). Nous décrivons ici un autre système de représentation. Nous l'avons employé avec des enfants de 1ère primaire (6 à 7 ans) pour représenter des points du plan et des figures géométriques à l'aide de coordonnées. Des notions telles la translation et la symétrie ont aussi été abordées.

1. INTRODUCTION.

In a previous paper (Lowenthal & Marcq, 1980), we have shown that the use of the formalism described by Cohors-Fresenborg (1978) can favour the development of structured communication among 7 to 8 year olds. One of us had already described the importance of another formalism and he generalized these observations to the use of several kinds of auxiliary formalisms (Lowenthal, 1977; 1980). We describe here another representation system. It is non ambiguous, simple and easy to use. It is also non-verbal but it involves the three Brunerian levels of representation : enactive, iconic and symbolic (Bruner, 1966). We used it with 6 to 7 year olds (1st graders) to represent points and plane figures (polygons) by means of a coordinate system. We show here how the children used this representation system to approach concepts such as translation and symmetry.

II. MATERIAL AND TECHNIQUE.

a) The children : we worked in a 1st grade class. There were 20 children, working by groups of 2. We did not let the same children work continuously together : we changed the "grouping" to enable smart pupils to explain their results to less gifted children. There were 10 sessions (one every week) and each lasted about 90 minutes. The first lesson took place on January 14 and the last one on April 1, 1981. Every lesson was video-taped.

b) The concrete support : we used an overhead projector. We had drawn on a transparency 121 dots, defining a squaring, as shown in figure 1. There were also "red numbers" for the y-axis and "yellow numbers" for the x-axis. We put other transparencies on top of the first one to draw polygons and

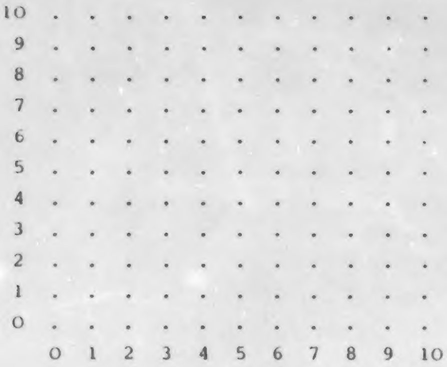

figure 1.

show them, using as points, only the dots of our grid. We were thus in fact working in the N x N plane. Each group of children had a small geoboard with 121 nails. This was a reproduction of our grid. They could also adapt a long red plastic strip, with the numbers 0 to 10 written on it, to materialize their y-axis. They also had a similar yellow strip for their x-axis. We gave them elastics to "join the nails" and represent the segments we would draw on transparencies. The children also got small pieces of multicoloured straws which could be fixed on a nail. These pieces were called "posts" and served to distinguish the points the children were talking about.

c) The technique : our research comprizes 5 periods or steps. (1°) We taught the children to associate coordinates to points and to mark with posts, on their geoboards, the points for which they had a name; (2°) we also taught them to use these coordinates to form messages describing polygons and to represent these polygons on their geoboards by posts marking the vertices and by elastics joining the two posts defining a segment or side of the polygon. Later we asked them (3°) to code and decode messages created by some of them and corresponding to polygons which were represented by elastics and posts on the geoboards. We tried (4°) to use these coordinates to approach the concept of translation and (5°) later that of symmetry : we asked the children to put on their geoboards a polygon with vertices marked by posts in one colour, and then to apply a transformation to the corresponding message. They had then to put on the geoboard the polygon corresponding to the new message, with vertices marked by posts of another colour, next to the original polygon. Finally we asked them to compare the polygon with posts in the "old" colour and the polygon with posts in the "new" colour.

III. RESULTS.

In a first step, we taught the children to "name" each point : they had to associate to each point its coordinates presented as an ordered pair of numbers. By mistake, with respect to usual mathematical conventions, we told them to use red first and yellow second (i.e. the y-coordinate first). All the children were immediately able to find the name of a point that was shown to them, but many needed 2 more sessions before being able to show a point for which they had only a name (ordered pair of numbers).

In a second step, we asked the children to reproduce (with elastics and straws), on their geoboard, a polygon they could see on the screen. We then associated to each polygon a "message", or "name of the figure", by giving the sequence of vertices of our polygon, in a given order. The polygon shown in figure 2 received the following name : (3,2)►(2,3)►(1,3)►(1,1)►(2,2). We have then drawn polygons on our transparencies, like the square shown in figure 3 in dotted lines. We asked the children to find a message for this figure. At first they came up with messages like : (3,1)►(3,3)►(1,1)►(1,3). We then took away the transparency with our square, we put a new one on top of the squaring, we wrote on it the child's message : he was supposed to make a telephone call and tell a friend which polygon we were looking at. We used the child's incorrect message to draw the continuous line (Z) and asked the children to compare it with the original picture (to do this we superposed the transparencies). The children concluded that this message was incorrect and tried other messages. Eventually a girl came up with a correct message and was able to explain to all the pupils that : "When you write a message, you must give the names of the points in the correct order".

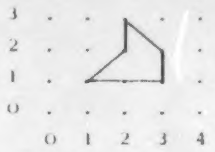

figure 2.

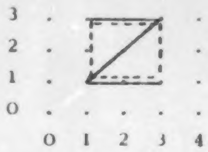

figure 3.

Many children (11 out of 20) were then able to create a figure for which they had nothing else than a message. In a third step, we asked them to "invent a nice figure with 5 posts and to write the corresponding message". They had then to give their message, but nothing else, to another group who had to reproduce the polygon invented by the first group. When the second group was ready, the first one came and checked : often, everything was correct, sometimes the first group had to help and suggest corrections. 11 children took part very actively in this "message game" during 90 minutes.

In a fourth step, we asked the children to put on their geoboards the polygon corresponding to the following message : (2,5)➡(3,2)➡(1,1). We asked them also to mark the points used by red "posts". 10 children did it easily and fast. We then put our own figure on the squaring and we told the children that "a friend gave us instructions by telephone : we had to do '+4 in red' ", we were supposed to write the new message and to realize the corresponding figure, marking the points with green posts. The children were active (especially 6 of them). They discovered that : "It stays on the same lines" (showing with the hands the vertical displacement). We asked them whether the new figure, with green posts, was the same as the original one, with red posts. After a short discussion they said that both were the same. They explained this by showing that the corresponding sides were equal, and also by saying that one could be put on top of the other. Nevertheless, the "+4 in red" operator created a problem as one of the children wanted to transform the original triangle (labelled (a) in figure 4) into triangle b : he was counting 4 dots, including the starting and final dots. Another child correctly suggested to create triangle c; she said : "I started at (3,2) and I did +4, so I got (7,2)". She then counted 4 intervals and explained the first child's mistake. Finally a third child explained that : "You get triangle b for '+3 in red' and you can pass from (b) to (c) by using '+1 in red' ". All the children noticed that the three triangles were "the same" and were "in the same columns". This first approach of the translation was restricted to translations parallel to the coordinate axis. It lasted 3 sessions and many participated.

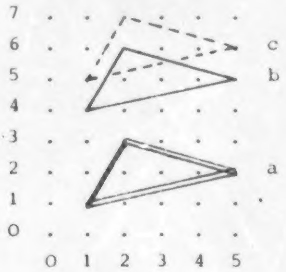

figure 4.

In a fifth step, at the end of a session and after a long discussion concerning translations, we put a transparency with a triangle on our grid. We asked the children to name this figure. One of the less active children gave the correct message. We then told the children that "a friend phoned and told us that all red numbers must become yellow, and all the yellow ones must become red", we asked the children to write the new message and to construct the new figure. In fact we were asking them to make a symmetry with respect to the y=x line. Writing the new message was not a problem; but its interpretation was more difficult : the name (4,6) becomes (6,4) and a child wanted to use this new name for the same point. We explained that the coordinate axis did not move and the children understood that the image of (4,6) by our red-yellow transformation is another point called (6,4). Only 12 of the 20 children were present and 4 were very active. They constructed the original and the new triangle on their geoboards, they said that these triangles were not the same "because you cannot put one on top of the other", but they explained, using their incomplete vocabulary, that : "they were the same in the sense that one is the reverse of the other". They used hand movements to show what happened. One child also said : "left becomes right, the lying lines (horizontal) become standing ones (vertical)".

IV. DISCUSSION.

This representation system is useful to teach young children what coordinate systems are; it is also useful to let them have a better grasp of the difference between left and right, above and below, horizontal and vertical, ...
We nevertheless met two major difficulties : within this representation system, a point is an ordered pair of numbers and, while a segment is only a set of two points, a polygon is an ordered and closed sequence of points.

The "exchange of message" session was unexpectedly successful. These non-verbal communication devices served as basis for fruitful verbal exchanges, firstly between the children who were creating the message, later between those who were decoding it and finally between all of them, when one group was checking the results obtained by the other. When this session ended, it became obvious that 11 children out of 20 had solved the difficulties mentioned above. Exercises about properties of the translation led to the discovery of the conservation of shape and to a first approach of parallelism. It was also a good opportunity to discuss the difference between "number of points" and "number of intervals between these points". As far as symmetry is concerned, we gave only a preliminary exercise and 4 children out of 12 gave the impression that they understood. They lacked the vocabulary but showed by gestures that both triangles would be alike if one could extract the original one out of the plane and turn it upside down onto the other. We intend to work further in these directions using translations which will not be parallel to one of the axis; we want to ask the children to work further with symmetries and then ask them to discover, whenever possible, which translation or symmetry we should use to bring one polygon (given by a message) onto another (given by another message). Finally we wonder whether it will be possible to let these children study the composition of translations and symmetries.

V. CONCLUSION.

This very simple representation system enables 6 to 7 year olds (1st graders) to approach easily, in a playful atmosphere, problems which are usually neglected by 1st grade teachers : the importance of coordinate systems and of ordering of points is obvious. But this should not let us forget the properties of translation and symmetry, which can thus be discovered by looking : conservation of shape and area (inaccessible to 1st graders ?), difference between a polygon and its image under a symmetry (why is my left hand a mirror image of my right hand ?). There is also the main characteristic of this system : the child learns first that points and polygons can be coded, they have a name which is easy to use; and then the child learns to use these names, to apply operators on "easy-to-handle" names instead of cumbersome objects. The child reaches then a first meta-level.

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THE ROLE OF DIAGRAMS IN MATHEMATICAL EDUCATION

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On fait la distinction entre des figures (qui communiquent des renseignements non-spatials) et des images (qui communiquent des renseignements spatials par des moyens spatials). On donne une classification des figures qui est fondée sur l'oeuvre de Plunkett, et qui est suivie par un rapport sur la recherche sur la compréhension des figures par les enfants. C'était une étude préliminaire; 125 enfants, âgés 12-15 ans, ont fait un examen écrit qui consistait de 45 questions (on devait assortir des figures et des histoires, interpréter des figures, et finir des figures). L'intention principale de l'oeuvre était la validation de l'examen, mais on donne aussi des indications de l'effet de l'âge, d'abilité et de sexe. On a fait une petite investigation sur l'emploi des figures en la solution des problèmes; on a fait aussi une analyse des figures dans trois cours de mathématique.

On discute des moyens de perfectionner l'enseignement des figures avec des indications des emplois possibles de l'ordinateur.

Diagrams as conveyors of information

Following Plunkett (1979), we distinguish between a 'picture' and a 'diagram'. Both are spatial representations of information - in the case of a picture the information is spatial, while for a diagram it is logical and non-spatial. Thus a map, a blueprint (plan) for building a house, a drawing of a motor car engine assembly are 'pictures', whereas an electrical circuit diagram, (only the connections, not the layout, are important), a critical path analysis for building a house, a graph of engine speed against time, are 'diagrams'. (Like all classifications this one probably has 'fuzzy edges' - is a triangle of forces a diagram or a picture?? It would be worth exploring these doubtful areas, but the main distinction will do for our purposes here.)

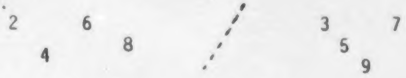
The main theme of this paper is that diagrams are an important means of communicating thought to others, and of holding thoughts in our own minds (ie an aid to thinking). They are a language which children need to master. Bishop's work is a striking illustration of the importance of cultural context in the way people handle pictures. Janvier has done extensive work on the interpretation of Cartesian graphs. However, as Plunkett remarks, we do not generally make any explicit attempt to improve children's proficiency in recognising, using and constructing diagrams - what they can do is almost an incidental effect of teaching. In the research which is reported briefly below, we tried to find out how well children understood diagrams, ie whether they could recognise and read them, and how, when encouraged to do so, they made use of them in solving problems. Using time-tables and understanding the number line, mentioned by Plunkett as a frequent source of difficulty, are aspects of mainstream school mathematics which we hope most children will master as part of their general education, and both belong within the classification of diagrams which we discuss next.

The classification of diagram-types

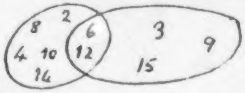
The classification discussed here, which is a slight modification of that due to Plunkett, involves seven distinguishable spatial relationships, which are used to display information. They are labelled (a) to (f) in what follows:-

(a1) 'Is near to' ('is in the same regions as', 'is inside'), usually used in classification.

Examples i)

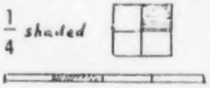


ii)



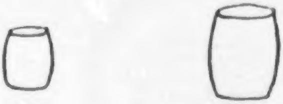
(a2) 'Is larger than' ('same size as')

Examples iii)



or (exact comparison of areas or lengths)

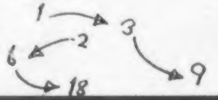
iv)



(approximate comparison of size (volume) as in advertisements)

(b) 'Is connected to'

Examples v) family tree
vi) flowchart
vii)



'is one third of'

used for general relationships

(c) 'Is next to' (1 dimensional)

Examples viii) 1, 1, 2, 3, 5, 8, 13, 21... (Fibonacci)

ix) 1, 4, 9, 16...

(d) 'Is next to' (2 dimensional)

Examples x)

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
...									

number square

xi)

(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
(3,1)	(3,2)	(3,3)	(3,4)	([illegible],5)	(3,6)
...					

dice scores

(in both (c) and (d), the relationships are within one set. The idea is that of sequence, or double sequence.)

(e) 'Is opposite' (1 directional) - a one-way table

Examples xii)

chips	20p
peas	15p
fish	90p

xiii)

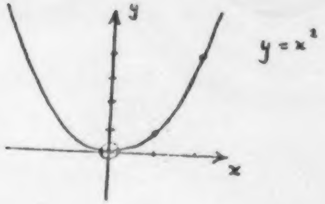
x	1	2	3	4
x^2	1	4	9	16

(f) 'Is oppposite' (2 directional) - a two-way table

Examples xiv)

+ mod 4	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

xv)



In (e) relationships between two different sets (in general) are indicated; the mapping described is $X \rightarrow Y$.
In (f) the relationships are between three sets, with mapping $(X, Y) \rightarrow Z$.

For the examples considered, in xiv) $X \equiv Y \equiv Z \equiv \{0, 1, 2, 3\}$, and in xv) we may take $X \equiv Y \equiv \mathbb{R}$, $Z = \{0, 1\}$.

Note, apart from the relatively unimportant (a2), all these categories are included by Plunkett - the chief difference here is the emphasis on the distinction between (c) and (e) (and (d) and (f)); it seems to me the underlying ideas are very different, though the common use of 'next to' in displaying them rather confuses the issue.

Neither classification caters for the nomogram(which uses the relation 'is collinear with'), though this might be forcibly included in (b) if a spurious "tidiness" is desired. (See Watson (1981) for a further discussion).

A report of research on children's understanding of diagrams

This exploratory study (Mandara, 1980) was carried out with a small sample of pupils aged 12-15 years in three schools. My purpose here is less to present 'results' than to discuss the nature of the study and the methods used.

i) A written test of 45 items was given to 125 pupils drawn from the first four years of three local schools. There were 69 boys and 56 girls; pupils were in 9 groups, six of which were described by the school as 'average', and three as 'above average'. This was an opportunity sample - groups were not matched or randomly selected and were, in fact, half-classes, (the other half of each class being involved in an unrelated investigation,so as to minimise disruption of normal school work). On the basis of written responses, a few interviews were carried out with 'interesting' subjects - these revealed, as anticipated, that wrong answers did not always indicate poor understanding of the diagrams, and that correct written answers could be arrived at by wrong reasoning. The main purpose of the investigation was to construct and try out the written tests, extracts from which are available. (A pilot version was tried out first in the normal way.) Since samples were small in size and possibly unrepresentative, statistical comparisons are inappropriate, so that the results obtained must be regarded merely as 'pointers' - they were that age and ability group had the anticipated positive effect on score, and that sex differences were not significant. A comparison between the two fourth-year groups, using SMP and SMG texts respectively, indicated that the textbook in use had no effect on scores - though a check showed that, with respect to the diagrams-test used, both series had covered fairly similar material.

ii) As an extension, a short investigation of the use of diagrams in problem-solving was made. Four problems were given to a group of children aged 14-15 years with specific instructions to "use a diagram as a stepping stone to the correct solution'. Some ignored the instruction - often giving a wrong answer, while others gave the correct answer, with an irrelevant diagram or picture which had clearly not been used to produce the solution; yet others did what was asked with good results. To avoid raising the distinction between 'diagram' and 'picture' (which had not been discussed with the pupils) a subsequent small-scale exercise with 27 pupils aged 13 years involved choosing the diagram (from a given collection) "which would be most helpful" in solving each of the 10 problems in the excercise. This revealed a number of misconceptions, and appears a promising avenue for further study. One feature which emerged pointed to the need to provide children with an adequate repertoire of diagram types - perhaps not surprisingly, they were unable to use or interpret diagrams which related to parts of the course they had not yet encountered. Diagrams are not self evident!

iii) A survey was made of the diagram-types encountered in three textbooks - the SMP 'letter' books (A - H), the SMG series (1-4), and New General Mathematics (Books 1-4). (The first two might be classed as 'modern', the third as somewhat more 'traditional'.) The three schools involved in the main study (i) used either SMP or SMG. New General Mathematics was used by a fourth school in which a pilot version of the written test (i) was tried out. This meant that comparisons between textbook series were restricted - since NGM appeared to place rather less emphasis on diagrams than the other two texts (particularly SMP), it would have been interesting to see whether this was reflected in the test results.

Venn diagrams (a1) and the number line (c). function tables and graphs (e and f) and linear programming (f) are common, as are statistical representation of data (frequency tables e, bar charts f).

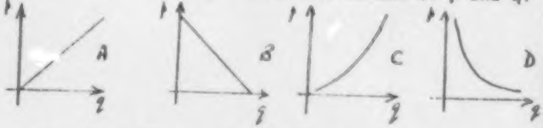
Flow charts (b) and arrow diagrams for relations (b) were also present, in SMP and NGM, while fraction - and dot number - diagrams (a2), distance-time graphs (f) and operations tables (f) were in SMP and SMG. What is less evident - and would repay careful study - is the extent to which the use of diagrams is an important feature of the course, and, more importantly, whether this is brought out in the actual classroom teaching.

Some general remarks and conclusions

i) For the most part, the questions were answered without too much difficulty. A few of the test items were taken or adapted from previous studies (Kerslake 1977, APU 1980) - facility values were generally similar to those obtained by earlier investigators. The interpretation of distance-time graphs, of (x, y) function graphs, and one question on generalised co-ordinates again caused problems. Some difficulties may have arisen from question ambiguity or lack of background, but low facilities in 3 questions were somewhat surprising:-

Q 10 (F = 8%, n = 95)

Jim has some numbers p and q which satisfy the relation p x q = 1 for all values of p and q.



Q 24 (F = 43%, n = 125)

The table shows bus fares from bus stop A to bus stops B, C... , H for adults. Find total fare paid by one adult travelling from D to G.

A	B	C	D	E	F	G	H
0p	15p	23p	34p	45p	58p	69p	80p

Q 43 (F = 18%, n = 125)

Towns A, B and C are on the same motorway (in this order). B is 6 miles from A and C is 9 miles from B.

Complete the table.

To / From	A	B	C
A	0	6	
B		0	

ii) Translation skills, from stories to diagrams and vice versa were involved. How can these best be developed? It seems that experience in using a repertoire of diagrams is likely to be helpful to children; it may be that in current courses their exposure to diagrams is inadequate to give them a confident mastery of this useful tool in problem-solving and communication. Can we improve on the heuristic advice 'Draw a diagram', by providing criteria for what sort of diagram might be appropriate? This links with the typology of diagrams which was attempted in the earlier part of this paper.

iii) The important topic of graphs has already received some deserved attention (Janvier 1978, Kerslake 1977). Some recent computer programs written by R J Phillips may indicate a useful way of introducing ideas in this area. Aspects of work with other types of diagrams would also probably benefit from similar detailed research studies, and the computer offers additional teaching possibilities in this area which merit exploration. There seems plenty of opportunity for work in this field.

iv) The computer may have an important role to play in other ways, too, for it can animate 2 dimensional representations, so that they change with time, and can also allow us much simpler ready access to 3 dimensional representations than that of building and manipulating models. 2-D views of apparently 3-D objects can be generated, and the 'object' rotated relatively easily. Sections across given planes could also be produced at will. This would quite literally 'add another dimension' to our thinking about diagrams.

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UNE APPROCHE EXPERIMENTALE POUR L'ETUDE DES PROCESSUS DE RESOLUTION DE PROBLEMES

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The aim of this paper is to present an experimental approach for research on problem-solving, which is different from those usually used. It was designed for a research on pupil proving activity which is, for reasons exposed in this paper, one of the main aspects of problem-solving processes.
This approach consists in putting pupils in a interaction and communication situation in which they have a common problem solving task. The main advantage of this experimental situation is that it enables the problem-solving process to be apprehended by means of the learner's formulation required by the problem solving activity itself.
After a presentation of this approach a description is maid of the experiment. The learners were 11 year olds (BALACHEFF 1980 a) and 15 year olds. A detailed example of a problem solving process is described.

Introduction

Nos travaux actuels sont centrés sur l'étude de ce que signifie prouver en mathématique pour les élèves de 11 à 16 ans (premier cycle de l'enseignement secondaire français).

L'objet de cet exposé est de montrer ce que l'outil expérimental que nous avons développé à cette occasion peut apporter comme moyens nouveaux au domaine de l'étude des processus de résolution de problème (Problem-solving) dont l'étude des processus de preuve est à notre sens un complément nécessaire.

Horizon méthodologique

Dans les lignes suivantes nous envisageons les principales approches expérimentales disponibles en signalant les limites qui nous permettront de nous situer et relativement auxquelles nous avons élaboré notre propre outil.

- L'épreuve papier-crayon ; elle permet difficilement de connaître les processus mis en oeuvre par l'élève. Il peut y avoir une distance importante entre le produit de son activité et cette activité elle-même (voir plus loin le cas de Pascal).
- Le "thinking-aloud" ; dans ces situations, on demande au sujet d'expliciter ce qu'il pense afin de rendre observable sa démarche de résolution. En fait il faut regarder la signification de ce qu'il explicite relativement au dispositif expérimental : le choix par le sujet de ce qu'il énonce peut être déterminé par ce qu'il perse être pertinent pour l'observateur. Ceci prend une importance particulière lorsque le sujet est un élève tandis que l'observateur est un adulte.

- L'interrogatoire clinique ; ici l'observateur intervient par ses questions ou ses suggestions, dans le processus de recherche. Il peut ainsi agir sur les choix du sujet, attirer son attention sur des faits qu'il avait négligés ou introduire de nouvelles informations, et donc modifier de façon profonde sa démarche de résolution.

Résolution de problème et démarche de preuve

Résoudre un problème signifie en trouver la solution, c'est-à-dire établir qu'un énoncé donné, ou que l'on a découvert, est vrai.

Cela signifie que dans le processus de résolution il y a une démarche par laquelle le sujet vise à s'assurer que ce qu'il produit est valide ; c'est ce que nous appellerons une démarche de preuve. L'étude de cette démarche ou même sa prise en compte est absente de la plupart des recherches.

Cette lacune conduit à ne retenir du processus de résolution que l'aspect traitement de l'information, en identifiant les outils mathématiques utilisés ou les thèmes heuristiques mis en oeuvre. Le risque d'une telle approche est de présenter ce processus comme séquentiel alors qu'il est en fait le lieu d'une dialectique entre la situation problématique (le problème et son environnement) et le sujet comme système de connaissance. L'analyse épistémologique d'I. LAKATOS (1976) donne un modèle du fonctionnement de cette dialectique en montrant comment, sous la pression des contre-exemples, peut évoluer la solution d'un problème.

Ainsi il nous apparaît que le processus de résolution est inséparable de la démarche de preuve, et qu'elle en est même un constituant. Si nous continuons dans la suite à les distinguer c'est que l'expression "Résolution de problème" (Problem-solving) a pris une signification très fortement lié au seul traitement de l'information.

Nous considérons la résolution de problème (Problem-solving) et la démarche de preuve (Proving-process) comme deux processus en interaction étroite dans l'activité du sujet résolvant un problème.

Une autre approche expérimentale

De même que pour les démarches expérimentales que nous avons évoquées, nous cherchons à atteindre les processus de résolution de problèmes mathématiques en provoquant des explicitations, en particulier des verbalisations. C'est à partir de ce matériau que nous essaierons de déterminer le champ conceptuel dans lequel est résolu le problème et d'identifier les procédures envisagées et mises en oeuvre (Problem-space).

Nous désirons que cette explicitation s'intègre dans l'activité de l'élève en prenant sa signification dans la démarche de résolution elle-même et non dans les attentes, qui lui sont étrangères, de l'observateur.

Ceci peut être obtenu dans une situation d'interaction et de communication où les élèves ont pour tâche la résolution en commun d'un problème.

C'est une situation de formulation (BROUSSEAU 1978) car les élèves ne peuvent satisfaire à la consigne sans recourir à la verbalisation ou à des représentations. La communication n'est possible que s'ils parviennent à s'accorder sur un langage commun. Ceci peut provoquer une dialectique de la formulation dans laquelle nous verrons se préciser les notions utilisées.

Par ailleurs, les élèves doivent s'accorder sur des critères communs pour accepter ou refuser une assertion. Dans une telle situation peuvent apparaître des conflits sur le choix d'une stratégie, sur une opération à réaliser, sur la signification d'un concept ... Ces conflits engagent les élèves dans une dialectique de la validation (BROUSSEAU 1978) et des processus de preuve au terme desquels ils doivent parvenir à un consensus sur la validité des choix qui sont faits.

C'est dans ces débats que nous trouverons la signification des processus observés : contre quelles alternatives ils sont choisis, quels arguments sont développés ?

Ces aspects sont ceux qui sont favorables à l'étude que nous envisageons, mais dans une telle situation d'interaction il peut apparaître des phénomènes socio-affectifs puissants susceptibles de déplacer le débat hors du domaine cognitif qui nous intéresse. En particulier dans la démarche de preuve des arguments d'autorité peuvent se substituer à une argumentation portant sur les contenus ; du fait d'un engagement affectif important, des élèves peuvent délibérement soutenir des propositions fausses ou refuser des énoncés vrais.

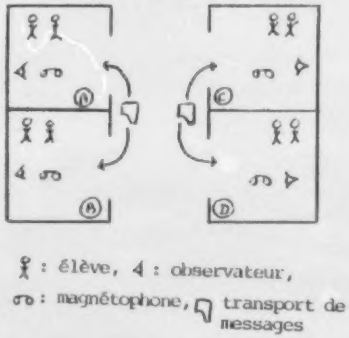
Un moyen d'éviter ces obstacles serait de régler la situation par un enjeu renforçant le désir de réussite des élèves, de façon à disqualifier ces comportements qui leur feraient courir un risque important d'échec. Nous en proposons une illustration dans le montage expérimental suivant :

Un exemple de situation

Nous allons décrire ici le dispositif expérimental que nous avons utilisé dans notre recherche :

Huit élèves sont répartis en deux équipes (AB et CD) de quatre élèves qui vont jouer l'une contre l'autre.

Dans chaque équipe les élèves sont séparés en deux binômes ((A,B) et (C,D)) qui chacun se trouvera dans une pièce différente ; ceci oblige à une communication par écrit, le but est de ralentir le processus pour en faciliter l'observation.


Ⓐ
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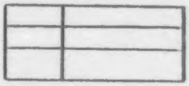
: élève, 4 : observateur,
: magnétophone, transport de messages

et d'obtenir des traces écrites des principales étapes de l'élaboration de l'explication.

Le problème posé est le même pour les deux équipes, la consigne est de le résoudre en donnant la meilleure explication possible.

Cette situation réalise une situation d'interaction et de communication telle que nous l'envisagions plus haut. Elle se présente ici comme une situation de jeu, l'intérêt qu'il y a à gagner constitue un enjeu suffisant pour ces élèves, âgés de 10 à 16 ans, pour obtenir l'effet que nous souhaitons : au sein d'une même équipe (il n'y a pas de communication entre les équipes) les élèves n'ont pas intérêt à soutenir des propositions fausses, ni à refuser des propositions vraies, une telle attitude les conduisant à perdre au jeu dans lequel ils sont solidaires.

Deux observations ont été menées avec ce dispositif, l'une avec des élèves de 6e (10-11 ans), l'autre avec des élèves de 3e (14-16 ans). Dans les deux cas le problème était de dénombrer les rectangles dans la figure ci-contre. (BALACHEFF 1980 b, 1981).



Le cas de Pascal

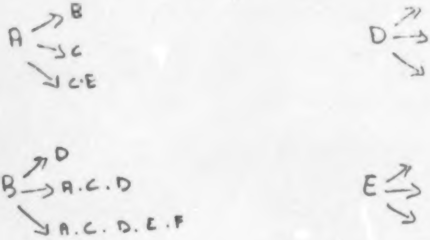
Le cas de Pascal est issu de l'observation conduite avec des élèves de 3e, il illustre la distance qu'il peut y avoir entre la production obtenue d'un élève dans une épreuve papier-crayon et la démarche qui y conduit. L'explication (reproduite ci-dessous) dont il amorce la rédaction ne rend pas compte, et ne permet pas de percevoir la richesse heuristique de sa démarche ; en particulier la présence de nombreux cycles au sens de GLAESER (1976).

Pascal et son compagnon Philippe dénombrent de façon empirique les rectangles dans la figure donnée, puis Pascal se demande *"Qu'est-ce qu'on peut démontrer là-dedans ?"* Pour cela il cherche à structurer l'énumération en l'associant à un procédé de décomposition ou recomposition de la figure.

Dans la suite Pascal abandonne cette approche de l'énumération et les raisons de ses tentatives apparaîtront dans ses débats avec Philippe. Il cherche une loi numérique en observant d'autres figures du même type que celle qui est donnée car il voudrait *"y démontrer par des chiffres"*.

En fait Pascal cherche une solution à laquelle il puisse donner un statut mathématique, il rejette l'énumération car dit-il : *"c'est bon quand on fait à la main"*. Ce statut est donné, à son sens, par l'usage d'outil mathématique ; le rapprochement de deux remarques lui permettent d'aboutir : la notion d'arbre d'énumération lui a été enseignée l'année précédente ce qui lui donne le statut d'outil mathématique, et d'autre part il note que : 6 x 3 = 18. Son objectif est alors d'assembler ces deux remarques. Il commence à le réaliser dans le texte ci-dessus en perdant de vue que le but est d'établir que 18 est le résultat, il ne parviendra pas au bout de cette tentative.

Nous avons 6 petits rectangle. Trouvant comme resultat 18 on
on en conclus qu'avec chaques petit rectangle on en fait 3 qui
seront chaque fois differents.



A	B
C	D
E	F

Il faut remarquer que le document produit par Pascal ne rend compte que des derniers instants d'une démarche de résolution qui a duré plus d'une heure et dont la majeure partie a été riche et cohérente du point de vue mathématique.

Quelques résultats

Nous rapportons ici les résultats de nos observations qui concernent l'analyse des démarches de résolution de problème.

En 6e comme en 3e le problème est abordé avec des procédures empiriques d'énumération, ces procédures évoluant au cours de l'expérience vers des procédures plus structurées. Mais alors qu'en 6e, cette structuration reste implicite, en 3e elle est recherchée explicitement et s'améliore sous la pression des exigences de la communication. On observe une interaction forte entre la mise au point de modes de désignation des rectangles et la structuration de l'énumération.

En 6e, l'incertitude sur le résultat est grande et conduit les élèves à des vérifications. Un obstacle à l'apparition de démarches de preuve est l'impossibilité des élèves à considérer la procédure comme articulation d'opérations élémentaires.

En 3e, les élèves ont rapidement la conviction d'avoir obtenu le résultat ; cela est renforcé par leur conscience du caractère systématique (qu'ils n'établissent pas) des procédures utilisées.

Les principales procédures observées sont les décompositions perceptives (ligne, colonne, petits pavés), les procédures de reconstruction de la figure à partir du rectangle et les procédures fondées sur la classification des rectangles suivant leur taille en nombre de rectangles élémentaires.

Remarques

Nous avons souligné les apports qui nous apparaissent essentiels de la méthode que nous utilisons. L'objet des débats lors du congrès sera, nous l'espérons, de mieux cerner ces apports et leurs limites. Parmi ces dernières, il en est une importante qui est dû au dispositif expérimental lui-même ; la démarche observée est celle d'un individu en situation d'interaction et est donc vraisemblablement différente de celle qu'il aurait suivie en étant seul, bien que ce que nous faisons apparaître appartienne à son comportement en situation de résolution de problèmes.

En sciences humaines, la plupart des situations expérimentales interagissent avec l'objet étudié et l'une des tâches du chercheur est de découvrir et contrôler cette interaction. Par ailleurs, la connaissance d'un phénomène passe par celle de ses divers aspects qui peuvent être saisis par différentes approches.

Nous pensons que l'outil expérimental que nous proposons apparaîtra comme complémentaire des moyens déjà disponibles.

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LA REPRESENTATION SYMBOLIQUE D'OPERATIONS ADDITIVES EN SITUATION D'INTERACTION ET DE COMMUNICATION *

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Our purpose is to lay the accent on methodological questions about the interpretation of symbolic representations, in additive problems. These representations were collected in situations of interaction and communication. The data will allow to debate the question of the statute of the symbolic representations.

Le but de cet exposé est de mettre l'accent sur des questions méthologiques d'interprétation des représentations symboliques. La présentation des données recueillies selon un schéma expérimental classique (Groupe Expérimental/Groupe Contrôle) dans des situtations d'interaction et de communication servira à poser le problème de la détermination du statut de ces représentations.

1 - Position du problème

Le problème didactique qui nous étudions concerne l'influence des interactions et communications entre de jeunes élèves (2e année de l'école primaire) sur la production de représentations symboliques dans la résolution de problèmes additifs.

1.1. Notre recherche se situe à l'intersection de deux courants :

- les travaux de la psychologie sociale génétique, qui, reprenant l'analyse de Piaget selon laquelle la "co-opération" est en corrélation étroite avec le développement de la logique, a développé un paradigme expérimental metttant en évidence le caractère causal de l'interaction sociale sur le développement cognitif. Ce facteur opère pour le mécanisme du "conflit socio-cognitif" (A.N. Perret-Clermont, 1979) ;
- les travaux en didactique des mathématiques : G. Brousseau (1978) distingue clairement les caractères de formulation dans les situations didactiques, en insistant sur leur fonction dans le processus d'acquisation.

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* Ces recherches ont été menées dans le cadre du contrat n° 1.706.0.78 entre A.N. Perret-Clermont, J. Brun et le Fonds National Suisse de la Recherche Scientifique en collaboration avec M.L. Schubaner-Leoni, El Hädi Sanda et F. Conne.

G. Vergnaud (1977), développant sa thèse de l'homomorphisme entre la réalité et la représentation, a bien montré, à travers l'exemple de l'acquisition de l'addition, la nécessité d'une distinction entre le concept (signifié) et sa représentation symbolique (signifiant), afin de mieux comprendre les rapports qu'ils entretiennent et comment ils s'appuient l'un sur l'autre.

1.2. Une connaissance mathématique intériorisée se manifeste chez un sujet par des procédures d'action et par des procédures de symbolisation (formulations orales ou écrites). La formulation d'une connaissance mathématique n'est pas la simple expression, comme allant de soi, d'une conceptualisation déjà formée. Elle a ses propres règles d'élaboration, en liaison avec la construction conceptuelle. Trop souvent la question didactique est posée en des termes tels que "le passage au symbolisme", et considérée comme découlant naturellement de la maitrise des opérations par simple association des symboles adéquats. Nous pensons pour notre part que l'appropriation de l'écriture symbolique se fait :

- au contact de la construction sociale qu'est le système des signes et leur syntaxe. Colette Laborde (1980) a bien montré l'importance de l'analyse historique de cet héritage culturel. L'élève qui s'y voit confronté s'en fait sa propre représentation, en fonction de ce qu'il est à même d'y investir, et de la situation dans laquelle il se trouve. Il ne peut se l'approprier d'emblée;
- par confrontation des points de vue avec les autres acteurs de la situation didactique aux prises avec la nécessaire formulation de la connaissance mathématique. Dans ce contexte social d'interactions et de communications s'élabore progressivement un code approprié ;
- en fonction des représentaitons que se fait l'élève des concepts à actualiser dans la situation. L'activité de symbolisation est liée au sens attribué aux exigences cognitives de la situation et à l'identification que l'élève est à même de faire de la connaissance en jeu.

Notre hypothèse de recherche est que la mise en situation d'interaction et de communication favorise, chez les élèves, la production de représentations symboliques, et permet l'évolution de ces représentations. A partir des recherches expérimentales mises en oeuvre pour vérifier cette hypothèse générale, nous aborderons, ici deux catégories de questions :

a) quelles classes de procédures de symbolisation peut-on identifier dans ces situations ?

b) quel est le statut des représentations symboliques ainsi actualisées ?

Dans une recherche précédente, M.L. Schubauer-Léoni et A.N. Perret-Clermont (1980) ont défini ainsi les situations d'interaction et de communication :

- "Interaction entre pairs avec communication à un troisième". Deux enfants codent ensemble un message en vue de le communiquer à un troisième, qui devra le décoder en présence de l'expérimentateur et des deux codeurs ;
- Interaction entre pairs : deux enfants codent ensemble un message que seul l'expérimentateur verra et sur lequel il n'émettra pas de jugement ;
- communication à un pair : un enfant code un message qu'un deuxième enfant devra décoder en présence de l'expérimentateur et du codeur".

2 - Méthode

Nous avons étudié l'activité de codage d'une équation (a+b-c = x) par des élèves scolarisés en 2e année primaire (7-8 ans).

- le dispositif expérimental se compose d'un jeu de 3 dés avec les règles du jeu suivantes : deux dés sont rouges et font gagner des points ; un dé est vert et fait perdre des points. Un support au jeu de dés est fourni sous forme d'un parcours de cases où l'élève, peut avancer, ou reculer un pion, en fonction des points gagnés et perdus en lançant les dés ;
- la consigne invite les élèves à composer les gains et les pertes (ou avances et reculs) pour obtenir le bilan du lancer de dés, et à représenter par une écriture symbolique les opérations effectuées. Cette consigne de codage est la suivante : "Vous marquez tout ce qui s'est passé avec les points pendant le jeu et les points que vous avez à la fin du jeu". Aux questions des élèves, l'expérimentateur se limite à répondre : "Comme vous pensez le mieux" ;
- le déroulement de l'expérience s'effectue en trois temps à quelques jours d'intervalle. On vérifie, préalablement au moyen d'une épreuve collective papier-crayon, que les sujets connaissent les symboles arithmétiques nécessaires (+, -, =).

a) temps 1 : situation Expérimentateur-Elève. Chaque élève code le lancer de dés selon la consigne ci-dessus.

b) temps 2 : un groupe d'élèves (1 classe) est réparti en duos, mis en situation d'interaction ; où la communication est seulement invoquée. Le dispositif expérimental et la consigne sont les mêmes. Deux codages sont effectués. Le groupe contrôle (une autre classe) ne passe pas à ce temps 2.

c) temps 3 : répétition de la situation Expérimentateur-Elève. On introduit une variante au dispositif : un seul dé est lancé trois fois : lors du premier lancer on gagne des points et lors du troisième lancer on perd des points.

3 - Analyse

3.1. Les classes de procédures de symbolisation

La variété des écritures symboliques obtenues nous donne une meilleure connaissance des systèmes de représentation (signifiants) utilisés par les élèves pour coder la composition des quantités positives et négatives, et de leur écart avec l'écriture conventionnelle (qui nécessite l'utilisation des signes +, -, =).
Les élèves utilisent :

- le dessin, avec représentation d'indices perceptifs, tels que la couleur ou le placement des dés pour symboliser les opérations ;
- le langage naturel, avec des expressions telles que "gagné, perdu ; avancé, reculé ; en plus, en moins ; en tout, ça fait, fin du jeu" ;
- les signes conventionnels : +, -, =

Cette connaissance des formulations est importante, mais, à elle seule, elle ne suffit pas. Il est nécessaire de mettre en relation cet aspect "signifiant" de la représentation avec l'aspect "signifié", c'est-à-dire avec les opérations de pensée liées au concept en jeu dans la situation. Alors seulement, selon nous, on pourra parler de procédures de symbolisation.

Nos observations ont mis en évidence trois niveaux de traitement du concept :

1. Aucune composition des quantités en jeu dans le lancer des trois dés, qui est seulement décrit.
2. Une composition partielle des quantités.
3. Une composition complète des quantités.

En croisant les aspects "signifiant" et "signifié" on peut alors catégoriser les productions écrites en termes de procédures de symbolisation.

3.2. Le statut des représentations symboliques observées

Quand on observe la répartition des productions écrites, recueillies dans nos recherches, on constate une certaine dispersion de ces productions. Ainsi, à l'intérieur du même niveau de traitement du contenu mathématique de la situation, différents systèmes de symbolisation sont présents.

En particulier, si l'on considère le niveau 3, où l'élève effectue la composition complète $a+b-c = x$, on voit que l'écriture symbolique de cette composition s'exprime, chez des enfants de cet âge, à la fois par des indices perceptifs (couleur, placement des dés ou des chiffres), par du langage naturel, et par des symboles mathématiques.
On aurait pu s'attendre à un degré de cohérence plus grand entre le niveau conceptuel et l'utilisation des symboles conventionnels connus des enfants. Or nous constatons des décalages importants. Ceux-ci nous interrogent sur le

statut de l'utilisation que font les enfants des symboles en question. Ainsi en est-il du signe = . Il convient de se demander quel sens lui attribuent les élèves quand on constate que la majorité du groupe de ceux qui ont effectué une composition complète des quantités en jeu utilisent l'expression "en tout", "ça fait", pour signifier le résultat d'un calcul, et non la relation d'égalité.

Méthodologiquement il nous semble insuffisant de fonder une hiérarchisation des productions écrites sur le seul degré d'adéquation à l'écriture conventionnelle, même si l'usage de ce code n'est pas indifférent. La question soulevée ici est celle de son statut dans l'usage qui en est fait.

Dans une recherche, menée parallèlement sur la lecture des égalités de type $a+b+c = x$, F. Conne a dégagé trois niveaux de signification pour les élèves :

a) Impératif : l'élève lit les signes comme un ordre qu'on lui donne ;

b) calcul : l'élève voit dans l'écriture la représentation de son action. Le nombre final est pris comme l'aboutissement d'un calcul et le = veut dire "ça fait".

c) Relation : lire la relation signifie alors qu'on suspend le déroulement du calcul. Le signe + par exemple ne signifie plus seulement une addition à effectuer, mais signifie l'opération elle-même. Le signe = accolé au résultat ne veut pas dire "ça fait" et il peut précéder les termes de l'addition. La concordance entre le déroulement temporel de l'action et le déroulement spatial de l'écriture importe moins.

Ces observations confirment, en même temps qu'elles éclairent, la nécessité de préciser le statut de signifiants produits. La situation dans laquelle évolue l'élève est constitutive de ce statut.

Le but de notre exposé était d'abord de soulever la question méthodologique de l'interprétation des productions écrites en termes de procédures de symbolisation, en proposant la mise en relation des catégories d'écritures symboliques avec le traitement des aspects cognitifs de la situation. L'analyse du rôle des caractéristiques d'interaction et de communication de la situation sur le statut des productions écrites est le second aspect de cette question. Elle concerne l'examen de l'évolution des productions. L'exposé de M. Guillerault et C. Laborde (Actes de cette Ve conférence) en souligne l'importance et montre le parti qu'on peut tirer de l'analyse du déroulement continu des interactions dans une situation où la variable temps est fixée sur une durée assez longue. Dans le cas de notre démarche expérimentale comparative, cette variable temps est neutralisée : deux codages uniquement sont demandés pour que les productions entre le groupe expérimental et le groupe contrôle soient comparables. Nous avons analysé l'évolution des productions aux différents temps (discontinus) de l'expérience. Les résultats de cette évolution montrent, sur de petits effectifs, une tendance à une progression plus importante des niveaux de procédures de symbolisation dans le groupe d'élèves mis en situation d'interaction.

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UNE SITUATION DE COMMUNICATION EN GEOMETRIE

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The aim of this paper is to analyse the linguistic procedures involved in a pupil activity, which consists in writing a mathematical discourse.
An analysis of mathematics texts reveals, in addition to the natural language, the usage of symbolic notations. Furthermore the natural language is extensively intermixed with the symbolic one.
Learners in the lower grades of the secundary school seem to experience certain difficulties not only in using the symbolic language of mathematics but even in designating mathematical objects by symbols.
An experiment was designed to analyse not only the formal features of the learners' discourse but also the cognitive processes, which generate their formulations.
This kind of study requires the construction of situations of communication in which social and cognitive aspects are involved. A situation of this type is presented and the verbal interactions between two learners are analysed. These learners have jointly to describe a given geometrical figure in a written message.
There is a discussion of features made relevant by this specific situation and comparisons are made with other approaches to similar problems.

I - Présentation des objectifs

Nous cherchons à déterminer les *procédés linguistiques mis en oeuvre par les élèves* dans une activité d'écriture d'un discours de type mathématique. Plus précisément, nos questions sont à placer dans le cadre suivant :
Les textes mathématiques écrits présentent une utilisation conjointe et parfois très imbriquée de deux codes : le code symbolique et la langue naturelle, indispensables tous deux non seulement pour l'expression mais aussi pour le développement des mathématiques.

L'élève de 11-12-13 ans est particulièrement confronté au problème de l'utilisation du code symbolique, car c'est à partir de cet âge qu'on lui demande plus systématiquement qu'auparavant de lire des textes mathématiques présentant cette imbrication des deux codes, de désigner des objets numériques ou géométriques par des lettres. Or, l'emploi d'un symbolisme est porteur de difficultés tant au niveau conceptuel (représentation des objets par des symboles) qu'au niveau linguistique (l'organisation du discours se trouve modifiée de par l'utilisation du symbolisme à l'intérieur même de la langue naturelle).

Comment les élèves de cet âge utilisent-ils le code symbolique et la langue naturelle et les font-ils fonctionner dans leur discours, pour d'une part désigner des objets, d'autre part exprimer des relations entre ces derniers ?
Quelle est la part de l'implicite dans leur discours ?
Telles sont nos questions, mais nous les posons non pour étudier les seules propriétés formelles de ce discours mais pour les appréhender en rapport avec le contenu véhiculé et avec le sujet parlant. Autant que les formulations écrites des élèves nous importe le travail cognitif qui a conduit à ces formulations. Nous rejoignons en cela les théories psycholinguistiques développées par M. Brossard (1978) ou par A. Culioli (1976). Les énoncés ne prennent leur sens entier qu'intégrés dans l'activité langagière, qui dépend de nombreux paramètres : ils sont résultats d'une construction d'un sujet énonciateur dans une situation donnée (qu'on appelle situation d'énonciation). Nous nous plaçons ainsi dans le cadre théorique dans lequel Jean Brun (1979) définit une approche en psychopédagogie des mathématiques. "... Cette étude des procédures et des représentations (il s'agit des signifiants) revêt toute sa signification en psychopédagogie des mathématiques lorsqu'elle prend en compte les caractéristiques de la situation didactique afin de préciser les conditions de production et d'évolution de ces procédures et de ces représentations".
Les questions que nous posons ne peuvent donc recevoir de réponses que relativement à une situation d'énonciation précise, dans un domaine conceptuel déterminé où seront pris les objets mathématiques. Nous avons retenu le domaine géométrique. Les objets mathématiques sont les points et les segments, les relations entre ces objets les relations d'incidence et les relations métriques "usuelles" (distance entre 2 points).
Nous avons construit une situation d'énonciation dans laquelle les formulations obtenues constitueront effectivement la solution à la tâche demandée aux élèves et ne seront pas simplement le résultat d'une rédaction dans un deuxième temps de la solution déjà élaborée auparavant.

A cet première exigence s'ajoute une contrainte sur le plan méthodologique : la nécessité pour nous d'avoir une trace de la genèse des formulations produites par les élèves.

II - Le dispositif expérimental

1) Description

Dans le cadre d'une situation de communication, une figure géométrique est proposée à un groupe de deux élèves (binôme des émetteurs ou codeurs). La situation proposée est celle du type "jeu de messages".

Chaque groupe de codeurs disposant d'environ 50 minutes a pour consigne de se mettre d'accord sur la rédaction d'un message ne devant comporter aucun dessin, destiné à des élèves du même âge (11-13 ans), et à qui il doit permettre de reconstituer exactement la figure donnée.

Un observateur note les principales phases de l'élaboration du message, et la discussion entre les deux émetteurs est enregistrée. Une fois le message terminé, si les codeurs n'en ont pas eux-mêmes déjà pris l'initiative, l'observateur leur demande de procéder à une vérification consistant à redessiner la figure à partir des indications de leur texte.

La situation d'énonciation que nous utilisons correspond à cette phase du jeu. Elle a été pratiquée avec 80 binômes pris dans des collèges de la région grenobloise.

2) Analyse

a) la tâche passe effectivement par l'activité de formulation :

La situation construite rentre dans le cadre des situations de formulation décrit et utilisé par G. Brousseau (1977), dans lequelles récepteurs et émetteurs sont engagés dans un jeu commun dont le succès dépend des deux partenaires, condition importante quant à la recherche du "meilleur" message par les émetteurs. De plus, les interlocuteurs même non présents étant des élèves et non le maître capable de comprendre des formulations peu claires, les émetteurs sont enclins à veiller à la clarté et à la précision de leurs formulations.

b) La structure de la situation :

Les caractéristiques importantes de cette situation d'énonciation sont les suivantes :

- la présence de deux locuteurs qui ont à rédiger un message commun pour un interlocuteur, et qui procèdent à des échanges verbaux pour réaliser cette tâche. Cette interaction entre les deux émetteurs est un facteur fondamental de notre situation. Elle permet d'obtenir une trace de la genèse du message, de provoquer l'extériorisation de l'analyse géométrique de la figure et des choix linguistiques qui ont lieu pour la décrire (choix d'un codage par lettres en nombres pour certains éléments clés de la figure par exemple). Le seul recueil de formulations écrites d'un élève ayant à effectuer exactement la même tâche ne nous aurait apporté aucune information sur l'élaboration de ces formulations.

  L'observation individuelle avec questions posées par l'expérimentateur serait susceptible d'apporter une information à ce propos. Nous préférons remplacer la relation interviewer-interviewé, qui peut constituer parfois un élément parasite, difficilement analysable, par une interaction entre pairs, dont A. Nelly PerretClermont (1980) et Jean Brun (exposé dans ces actes) montrent l'importance sur le plan cognitif.

Citons ici quelques aspects du rôle joué par cette interaction dans notre expérience :

- facilitation dans l'avancement du travail, par la possibilité de répartir les tâches (mesures, comparaisons, écriture du message, tracés ...), par la présence du pair qui peut aider à surmonter les périodes de découragement et d'absence d'idées, périodes délicates à gérer par l'expérimentateur dans l'entretien individuel.
- une meilleure prise de conscience de l'état du travail, une meilleure évaluation de l'efficacité des activités entreprises ou en projet, grâce aux réactions critiques voire à la contradiction apportée par le partenaire, une anticipation possible sur les effets du message , par la présence du camarade qui peut évoquer l'interlocuteur absent.

Nous voyons dans cette interaction un élément susceptible de contribuer d'une certaine manière à la richesse des observations. La contrepartie en est que les données recueillies seront en général plus complexes à analyser que celles obtenues au cours d'un entretien individuel. Cependant, il serait abusif de conclure que notre méthode apporte nécessairement davantage d'informations que cette dernière. Le travail à deux peut comporter des phases non explicitées, il tend à étouffer certains modèles géométriques ou linguistiques et à en renforcer d'autres. (l'apparition du codage serait plutôt favorisée par le travail à deux). Nous sommes donc contraints pour une meilleure vue d'ensemble à un nombre d'observations plus élevé.

- l'interlocuteur est absent, et la possibilité n'est pas donnée aux décodeurs de manifester leur réactions d'incompréhension du message aux émetteurs. Ce feed-back, noyau de régulation de la situation n'existe pas dans notre expérience. C'en est une limite. Cependant, le temps assez long (variable de la situation) laissé aux émetteurs permet en fait, comme nous l'avons souvent constaté, une évolution importante incluant la prise en compte des décodeurs (communication invoquée).

c) Deux facteurs de la situation :

Après avoir fixé la structure de la situation, nous en avons déterminé deux facteurs importants parmi tous ceux dont elle dépend : la complexité de la tâche et la longueur du temps donné aux élèves pour leur travail.

Nous avons choisi une tâche complexe et donc laissé un temps assez long (de 50 à 60 minutes) pour l'élaboration du message.

La complexité de la tâche tient à la complexité de la figure proposée. Cette figure ne comporte aucun symbole (lettres, nombres, ...) accompagnateur ; nous voulions laisser les élèves libres de coder certains éléments. La structure de la figure, rend nécessaire la prise en considération de relations de dépendance entre les objets géométriques à décrire. Cette complexité va rejaillir sur la complexité du message verbal, particulièrement en cas d'absence de codage, ou en cas de présence d'un codage peu adapté à la procédure de description de la figure.

Nous faisons l'hypothèse que le temps important laissé aux deux codeurs leur permettrait d'élaborer de nouvelles stratégies quand ils prendraient conscience de leur inadéquation et que le jeu de l'interaction entre les deux codeurs pourrait se dérouler pleinement et plus subtilement que si cette interaction s'exerçait sur une durée réduite. Cette hypothèse a été confirmée par l'expérience, tant au niveau de l'évolution des procédures de description de la figure et des formulations écrites qu'en ce qui concerne l'interaction entre les deux élèves. Ainsi la réaction à une contradiction apportée par le partenaire n'est-elle pas nécessairement immédiate mais peut se produire plus tard au cours du travail à deux.

En conclusion à cette brève analyse de la situation construite, nous insisterons sur le caractère complexe et évolutif qu'elle présente et qui est le plus adapté, pensons-nous, à répondre à nos questions sur les conditions de production de formulations des élèves.

III - Illustration de la méthode : quelques données de l'expérience

L'étude des messages et des échanges oraux entre émetteurs confirme le fait que les productions obtenues dépendent pour beaucoup de l'interaction des deux émetteurs et de la longueur du temps de travail :

- dans la plupart des cas, une évolution très nette se produit dans l'analyse de la figure et donc dans la procédure de description. Une étape transitoire de description "inventaire" sans relations de dépendance entre différents éléments de la figure se fait jour au début du travail de très nombreux binômes ;
- en règle générale, si le codage est évoqué quelquefois d'emblée au vu de la figure, il est en fait beaucoup plus souvent adopté, comme solution de problème géométrique, linguistique (problème de vocabulaire ou d'utilisation de systèmes de renvoi). Solution pour un des émetteurs, il n'est pas toujours perçu comme tel par l'autre et dans bien des cas sera adopté sous la forme proposée ou sous une autre forme par les deux élèves après qu'ils se soient heurtés à plusieurs obstacles qui n'ont pu être surmontés.

En outre, la situation nous permet d'analyser d'une part l'élaboration des formulations sur le plan linguistique (ambiguïtés, vocabulaire), d'autre part, les significations accordées par les élèves aux notions géométriques en jeu (point, segment et relations d'incidence).

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PROBLEM SOLVING PROCESSES OF UPPER ELEMENTARY SCHOOLCHILDREN

A CASE STUDY

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Ce papier décrit une série d'interviews accordées à une élève inscrite au cours élémentaire pendant qu'elle résolvait qu du moins essaysit de résoudre une série de problèmes de mathématiques. Pendant les deux dernières années près de 90 élèves ont participé à ce projet basé sur les travaux de Polya et Kruteskii. Le programme est établi dans le but d'enseigner aux élèves des "5th and 6th grades" les procédés de solutions. Un choix d'élèves a été fait selon leur capacité (supérieure, moyenne ou inférieure) et ils ont été interviewés individuellement et périodiquement pendant plusieurs mois. On a exigé qu'ils réfléchissent à haute voix pendant qu'ils s'efforçaient de résoudre les problèmes. Les problèmes ont été choisis pour les structures fondamentales de mathématiques et aussi pour les modèles et leur relation avec le travail effectué en classe. L'élève choisie comme sujet de ce papier est en "6th grade" et selon les tests qu'elle a passés en 5th grade, serait d'une intelligence inférieure à la moyenne. Dans ce papier on présente ses réponses aux 3 problèmes apparemment différents mais qui en réalité exigent tous les 3 le nombre de combinés de deux éléments d'un ensemble donné. Les réponses montrent sa façon d'utiliser les procédés employés pour résoudre les problèmes et l'attention portée à la structure; elles indiquent aussi un niveau de capacité mathématique non apparente lors des tests donnés en classe.

For the past two years, approximately ninety fifth and sixth grade students have participated in a National Science Foundation funded project designed to investigate problem-solving processes of upper elementary school students. The project, based on the works of Polya and Kruteskii, consists of two components; an instructional component designed to teach problem solving and an observational component designed to study selected individual students as they solve or attempt to solve various mathematical problems in interview settings. As part of the observational component, students were interviewed every few months and asked to think aloud as they worked on problems. The problems were

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selected because of their underlying mathematical structures and/or patterns and their relationship to the classroom instruction. Descriptions of each of these components along with brief reviews of related research have appeared in earlier papers, (Silver, Branca and Adams, 1980) and (Branca, Silver and Adams, 1980).

This paper describes a series of interviews conducted with one student, a current sixth grader. The student has been involved in the program since she was in the fifth grade where she was evaluated as slightly below average in mathematics for the class, based on school administered tests. The student was considered to have excellent reading ability but did not achieve as well in mathematics compared to others at the same level. Three interviews will be considered, each dealing with a different problem that basically asks to solve for the number of combinations of two elements from a given set. The interviews were conductd during the seventh and ninth months of the fifth grade academic year and the sixth month of the sixth grade academic year.

In the first interview under consideration, the student was asked to solve the following problem:

Suppose there are 8 people at a party. If each person shakes hands with everyone else at the party, how many handshakes will there be?

Exerpts from the taped protocol of the student, S, and interviewer, I, appear below:

S Let's see - 8 people - that'll be probably 8 times 8 which is 64.

I Why do you think 64? How did you do that?

S Each person shakes hands with 7 people so that would be 8 times 7.

I Why did you change it from 8 times 8 to 8 times 7?

S 'Cuz at first I thought it would be -- I thought each person would shake hands with himself...but I forgot to realize that when they shook hands they'd shake hands with 7 people. They don't shake hands with themselves.

I OK Now - think about it again. 8 people, each shake hands with 7 people you say - so that's 8 times 7 is 56. Let's look at a different problem. Suppose that you and I are at the party. Let's say we shake hands. How many handshakes is that?

S One.

I OK - you shook hands with me - and I shook hands with you.

S Oh! That would be two handshakes! It would be one because me shaking hands with you and you shaking hands with me is the same thing. And so it would be counted that way, so it would only be - oh.

I Does that change the problem now? What are you thinking about now?

S Each person would shake hands with other people so probably be half as many as before so it would be like - oh darn! it can't be divided. It would be like each person would shake hands about -- each person would shake hands 7 times but their handshaking - one handshake would like count for two people so therefore --- I don't know what to do!

I OK -Suppose we do this, If you have a problem like this and you don't know what to do, a good strategy to use -- is to get a similar problem that might be a little bit easier. Let's suppose there are only 3 people at the party. Do you think you could do that problem?

S Uh -huh.

I Another good strategy is to actually make a model of this or to think of a model. .... What are you doing?

S Drawing stick figures. Stick figure 1 shakes hands with stick figure 2 -- and I draw a line and show 'em shaking hands ...so that's 2 handshakes and 2 and 3 here to shake hands.

I So the total would be what?

S Three. So maybe 8 would be 8 handshakes.

I OK. That would be a good hypothesis. If 3 is 3 handshakes, maybe 8 would be 8 handshakes. What are you going to do now? What are you thinking?

At this point the student drew 8 figures and drew lines to represent handshakes. She then switched to circles to represent the people and finally started to keep track by tallying.

After the student determined the answer to be 28, the following dialog took place:

I 28 - Now explain to me what you did again?

S I wrote the numbers 1, 2, 3, 4, 5, 6, 7, and 8 and draw a tally mark for each one they shook hands with. Then after they had shaken hands with everyone I erased them.

I OK. What if there were ten people at the party? How could you do that?

S The same way.

I Which would be what?

S By tallying each handshake or recording it or remembering it. It would take a lot of brains to remember it, though.

I In fact drawing the figures seemed to help you to do it.

S The lines get all messed up. Like a spider web or something.

I That was a good idea to make a tally mark. That kept track of all of those. Remember the first thing you thought about?

S Multiplying.

I What did you multiply first?

S 8 times 8.

I But then you changed that. Do you remember why you changed that?

S Because each person only shakes hands with 7 people.

I But then you did a different multiplication. Do you remember? What did you do then?

S 8 times 7.

I Right. Which is what?

S 56

I OK. But then you thought that wasn't right either. Do you remember why?

S Because you mentioned how about trying...because I got kinda confused and couldn't do it...and then you mentioned trying to do a problem similar to that but you know like a smaller number - and I tried it with 3. And that turned out to be 3 - and so that made me think of it.

I That made you think of doing what? That made you think of a different way?

S Uh, huh - a different way. And that one - the diagramming was OK because I just had to draw 3 lines.

I OK. Now also though we said something about when you did 8 times 7. Why was that wrong, 8 times 7? Do you remember?

S Because when people shake hands with each other like if one shakes hands with two, or A shakes hands with B or John shakes hands with Mary or whatever shakes hands with whatever, it's only one handshake. And you made me realize that when you had me shake hands with you.

I OK - but then you said something about 56 wouldn't be right. What did you say after that?

S Because it would only count as one if two people shook hands with each other.

I Do you remember you had an idea after that?

S Drawing figures and lines.

I OK - but there was something else you said after that. It had to do with counting each of the handshakes twice.

S Divide by two - but 7 won't divide by two.

I What did you want to divide by two?

S 56. Maybe..that would be smarter.

I And what would happen if you divide 56 by two?

S I'll find out -- I got 28!!

I What do you think of that?

S It's the right answer.

I The right answer. So you had a good idea before if you divide by 2.

S I just thought of dividing 7 and 8 by two and I thought "what?"

I Do you remember in class what we were saying - in fact just yesterday - if you're multiplying two numbers together and you want to divide by two, you only divide one of them by two, not both of them.

After the students confusion regarding division was cleared up, she was asked again about the problem with ten people at the party. She realized that in addition to using the tallying method, she could determine the answer quickly by multiplying 10 and 9 and dividing by 2.

The second interview under consideration took place two months after the first. The student was presented the following problem chosen because of its relationship to the handshake problem:

> Suppose there are 8 cities. If each city is to be connected directly to every other city with a road, how many roads would there be?

The student read the problem aloud and began making a sketch.

S Let's see... make a city... looks like a box... because I think maybe I can do it with a diagram...But then, I did the same thing with... people were supposed to shake hands, and that didn't work, because there was just a mess of lines all over... Oh, well, I'll try it out anyway... 1,2,3,4,5,6,7,8. That's 8 cities, because each box represents a city. Let's see; A,B,C,D,E,F,G,H, now, okay,... A is connected to F... oh! directly? Does that mean they have to go a straight line?

I No, not on your diagram. It doesn't have to be a straight line. What it means is that you can go from one city to another without going through some other city.

S Oh, I see... Well, when its like this... I have a city A, a city B, and then City C. City B can get to City C by the same road... but would it have to have a separate road?

I How are you thinking about it?

S I'm starting at City A. Then I'm going to a city past here, and they can go past there, and there, that cars can go... go like that, and like that, too. But that depends on how they're arranged. 1,2,3,4,5,6,7,8. Okay, there's 8, and each city has to have a road to each 8 of them. That's almost like that problem we were doing which is with the license plates, see how many combinations you can... Okay, and first... and this is almost the same thing. You can put, like, there's... you do it for each road. There's an A-B road, an A-C road, an A-D road, an A-E road, an A-F road, an A-G road, an A-H road. And so, then you go to B. There doesn't need to be a B road... a B-A road, because there's an A-B road. So then we'll go... there's no B-B road either... and... a B-C road, a B-D road, a B-E road,... and a B-F road, a B-G road, and a B-H road. Next is C.

The student continued systematically in this way until all 28 possibilities were listed.

The third interview in the series occured eleven months after the first. The following problem, also chosen because of its similar structure, was presented:

Suppose there are 10 students competing in tennis. If each student must compete with everyone else once. How many different two-person matches would there be?

The student's first response was to multiply 10 times 9 getting 90. She then evaluated that answer and stated, "for each two-person match it would probably be half of 90 or around there." She was not confident of this method and she immediately drew lines to represent the tennis players and proceeded to count.

S This guy has a two-person match with him, so he can't have any more two-person matches with that guy, and then he has one with him... that's 2,3,4,5,6,7,8. Eight? Oh, yeah, 4 and 5 is 9, so he has 9 matches. So no one else can have any matches with this guy. So that's... put down 9, okay, and erase that guy, because he's had all of the matches he can have. So then this guy has a match with everybody, so that's 1,2,3,4,5,6,7,8... that's 8 matches, Bye-bye... Say he loses all of them, he's kicked out. So this guy has all the matches he can, 7, erase him... This guy has all the matches he can, 8... woops, that's not 8, it is 6. Yeah, that's all the matches he can have, and this guy has all his, that's 5,... erase him... this guy has all his, 4,... and then 3, and then 2, and you can't get any more because there's no 2-person matches with just one person. And this is the winner. Her hair is standing on end because she put it that way because she didn't want to get real hot with her hair going down her shoulders. She's wearing "Nikey's". She has two arms. And she's real happy. So if you add all these up... 44.

When the student was questioned regarding her original thinking about half of 90, she could not rectify the two answers. The student was then asked if she had ever done a problem like this before. She remembered doing a problem the year before about "shaking hands" and remembered drawing lines and getting "tangled up".

The complete protocols of the interviews contained rich illustrations of both sophisticated mathematical thinking as well as serious misunderstandings. They illustrate the student's use of problem-solving processes and the fluctuation between attention to detail and to underlying structure. The interviews are time consuming to administer, but can provide a picture of a student's mathematical ability not apparent from classroom measures.

ANALYSIS AND IMPROVEMENT OF SIXTH-GRADERS ABILITY TO SOLVE WORD PROBLEMS

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Les deux objectifs de cette recherche étaient : (1) obtenir une meilleure compréhension des processus cognitifs qui se déroulent chez des élèves du sixième année de l'école primaire pendant la solution de problèmes arithmétiques; (2) examiner la possibilité d'améliorer au moyen d'instruction la capacité des élèves à résoudre des problèmes. Dans la première partie de l'expérience des données quantitatives et qualitatives concernant les comportements des élèves pendant le processus de solution de problèmes ont été rassemblées dans deux classes. Les résultats ont montré des défauts importants dans les stratégies de solution des enfants, surtout sur le plan de l'analyse des problèmes. L'hypothèse a été formulée que ses défauts peuvent être surmonter par une instruction appropriée. Afin d'éprouver cette hypothèse un expériment formatif a été fait. Un programme d'instruction a été appliqué pendant deux semaines dans la classe expérimentale. L'objectif du programme était d'instruire aux élèves une méthode pour résoudre des problèmes, dont l'estimation de la solution avant de procédér à l'effectuation d'opérations arithmétiques est le composant essentiel. On est parti de la supposition que l'estimation de la solution remplirait une fonction heuristique dans le processus de solution de problèmes, dans le sens que l'action d'estimer incite les élèves à analyser le problème et à vérifier la solution trouvée. A la fin du programme d'instruction un posttest a été donné dans la classe expérimentale, ainsi que dans le groupe de contrôle. Les résultats quantitatifs de cet expériment formatif supportent l'hypothèse, mais des recherches ultérieures sont nécessaires afin de pouvoir interprêter ces résultats en termes qualitatifs relatifs aux processus d'apprentissage.

ESTIMATING THE OUTCOME OF A TASK AS A HEURISTIC STRATEGY IN WORD PROBLEM SOLVING

The main objectives of the present study were to get a better understanding of children's solution processes with respect to arithmetic word problems and to investigate the possibility of improving their problem-solving ability through instruction. The view of problem solving behind this study can be summarized as follows. Children's ability in solving arithmetic word problems is strongly determined by the degree to which they have an appropriate orientation basis which enables them to approach unfamiliar tasks for which they lack a ready-made solution procedure intelligently and systematically. Possessing such an orientation basis involves pupils' being equipped with two complementary types of actions : (1) actions that consist of being able to use and apply relevant conceptual knowledge of subject-matter content such as concepts, principles, etc.; and (2) thinking procedures for analyzing and transforming a problem to the point where it has reached a form that is familiar and makes contact with specific subject-matter content (De Corte, 1980; De Corte & Verschaffel, 1980).

With respect to pupils of the sixth-grade, the highest class of the elementary school in Belgium, it is often established that they are not very successful in solving word problems. It was hypothesized in our study that this is mainly due to a lack of the second kind of actions mentioned above - namely, thinking procedures. Therefore, to improve sixth-graders' problem-solving ability they should acquire the attitude and the skills to analyze and represent the relations between the data of the problem before starting to perform computations. Besides techniques for problem analysis, verification actions form another component of the equipment of an efficient problem solver. We thought, then, that systematically estimating the outcome of a word problem before working out the solution would be an effective heuristic strategy that leads pupils to analyze the problem on the one hand and to anticipate the solution on the other. The analysis of the problem provides the problem solver with an appropriate orientation toward the solution process, while the anticipated solution provides him with a means for verifying his final outcome. We have defined the concept estimating as follows : estimating is trying to get the approximate solution to an arithmetic task by passing roughly and in an abbreviated way through the solution process.

ASCERTAINING STUDY

Within the theoretical framework described above, we carried out an investigation consisting of an ascertaining and a teaching experiment (Kalmykova, 1970, p. 128; see also De Corte & Verschaffel, 1980). The objective of the ascertaining study was to determine how well (performance data) and how (process data) sixth-graders solve simple and more complex word problems. A specially designed test was administered in an experimental (N=20) and a control class (N=21). The test consisted of ten items : one numerical task (multiply .523 by 289.25) and nine word problems. Four of the nine word problems were of the more simple type : for example, "Maria got 180 fr. (francs) to go to the bakery; she received 11.25 fr. in change; how much did she have to pay ?" The other five word problems were more complex :

for example, "Five workers got their wages after ten days; altogether they received 50,000 fr.; under the same conditions, how much would the total amount of the wages of six workers after eight days be ?" Besides the test scores, we collected more qualitative data on the experimental pupils' actions during the solution of some items. The techniques used for this were written reports of the 20 pupils describing their solution processes with respect to two problems, and thinking-aloud protocols of 3 children with regard to three problems.

We found that, on the average, these sixth-graders commit a great many errors, namely 53 % in the experimental class and 42 % in the control group. On the other hand, there is a significant difference between the two groups ($p < .05$); in spite of this, we have, nevertheless, continued the study with those classes. When comparing the results on the test administered at the end of the teaching experiment it will, of course, be necessary to take the difference in initial level between the groups into account.

In a further analysis, two categories of errors were distinguished : thinking errors due to choosing and carrying out an incorrect arithmetic operation, and technical errors due to mistakes in the execution of an arithmetic operation (De Corte & Verschaffel, 1980). The distribution of all the observed errors in the experimental class over the two categories is as follows : 78 % thinking errors and 22 % technical errors. This result justifies the conclusion that, for those sixth-graders, the difficulties with respect to word problems are set primarily in the thinking phase of the solution process. We hypothesize that this is due to the fact that pupils do not sufficiently apply methods for problem analysis to more or less unfamiliar tasks. The most important result of the analysis of the qualitative data is that sixth-graders, in fact, employ systematic analysis rather rarely when they are confronted with a word problem. On the contrary, it seems quite customary for them to start performing computations almost immediately after they have read a task, or they try to get external cues - for example, by asking questions - concerning the computations that have to be performed. Another finding is the almost complete lack of verification actions performed by pupils.

We can conclude that sixth-graders are not very successful in solving word problems, and our findings support the hypothesis that is mainly due to the fact that those pupils do not possess the attitude and the skills to analyze problems.

TEACHING EXPERIMENT

The teaching experiment was an attempt to contribute to the verification of the following hypothesis : fostering pupils' thinking skills by equipping them with appropriate methods for problem analysis will lead to an improvement in their ability to solve arithmetic word problems. To stimulate skills in problem analysis among the experimental learners, they were taught a solution procedure, the core of which consisted of the use of estimating as a heuristic strategy. More specifically, we developed a five-point solution procedure : (1) read the task; (2) estimate the solution; (3) solve the task; (4) verify the solution; (5) note the solution. The experimental teaching program was implemented during a two-week period in the experimental class. Meanwhile the control class was taught according to the normally prescribed arithmetic program : the teacher presented tasks which were similar to those discussed in the experimental class though treated in the usual way - namely, without instructing the pupils systematically in the heuristic estimation strategy. When the experimental teaching program was terminated, a posttest was administered to the pupils of the experimental and control groups. The test consisted of two parts : the ten items of the pretest (part 1) and ten new items (part 2), similar in nature to the pretest tasks. The pupils of the experimental class were again asked to write a short description of the solution process employed in the same two items as in the ascertaining study.

Table 1 gives an overview of the average results of the two groups on the pretest (i.e. the test of the ascertaining study) and the posttest. In addition to the data in the table, we mention that, on the posttest, there are no significant differences between the experimental class and the control class; we reiterate here that the pretest scores were significantly better in the control group.

Tabel 1. Average results (in %) in the experimental and the control groups (1)

	Pretest	Posttest, part 1	Posttest, part 2	Posttest, total
Experimental class (N=20)	47%	65% (**)	76% (**)	71% (**)
Control class (N=21)	58%	55% (n.s.)	70% (**)	62% (n.s.)

(1) For each of the posttest results, an indication is given whether there is a significant difference with the pretest score : ** = significant at the .01 level (t-test); n.s. = not significant.

Table 1 shows that the result of the experimental group on the posttest is significantly better than on the pretest. The result of the control group on part 1 and on the total posttest is at the same level as on the pretest, while there is a significant increase on part 2. This last finding means that part 2 of the posttest was probably easier than part 1; the difference between parts 1 and 2 in the experimental class points in the same direction. From all these data, we can conclude that the findings support the hypothesis which was the starting point of this teaching experiment : when we teach pupils a solution procedure for word problems in which estimating as a heuristic strategy is of prime importance, their ability to solve such problems will increase.

The direct comparison of the experimental and control classes is thwarted by the significant difference between the initial levels of both groups. Nevertheless, the comparison leads to findings that are convergent with the preceding conclusion. The significant difference between the two groups established on the pretest in favor of the control class, no longer occurs on part 1 of the posttest; there is even an obvious tendency in the opposite direction. Indeed, the score of the experimental class is here 10 % higher than in the control class, and this difference is almost significant at the .05 level. The score of the experimental group on part 2 of the posttest is also higher than the result of the control group; the same is, then, true for the total test. However, none of these differences is significant.
A more detailed analysis showed that the score of the experimental pupils on the simple word problems in the posttest (part 1) is 21 % higher than on the pretest; this difference is significant at the .01 level. For the complex word problems the difference is 15 %, which is just below the .05 level of significance. In the control class pupils performance on the posttest for the two types of problems lies at the same level as on the pretest. In other words, in contrast with the control group, the experimental class makes considerable progress on both types of word problems.

We attempted to collect qualitative data concerning the processes and actions underlying children's problem-solving performance on the posttest by means of an analysis of their answer protocols and of the written reports. However, we have not been very successful in this regard, as compared to the ascertaining study. The quantitative results discussed above show that the children of the experimental group, who have acquired a solution strategy during the experimental teaching program, achieve better results in solving arithmetic word problems. Undoubtedly this has to do with the fact that, by applying the solution procedure, they are more appropriately oriented toward solving the tasks. The data collected do, however, not allow us to decide whether or not the quantitative improvement in achievement is due to the acquisition of the heuristic estimation strategy as such; further research is needed to examine if this is the determining factor in the solution procedure. Meanwhile, in view of such research, we wish to state the hypothesis that the acquisition by the learners of the heuristic estimation strategy leads to a qualitative improvement in their problem-solving activity with respect to word problems, through which a quantitative increase in achievement occurs. This hypothesis is not only based on the central position of the estimation-strategy in the solution procedure taught to the children but also on certain data that emerge from an analysis of the scores and the answer protocols from the posttest.

In further research with respect to this hypothesis, special attention should be paid to the collection of qualitative data on pupils' problem-solving processes before, during, and after the experimental teaching program. It is also our opinion, however, that, in view of theory-construction about learning to solve problems in elementary school children, it is extremely desirable as well to conduct, in addition to classroom teaching experiments, clinical teaching experiments in which the experimental teaching program is implemented with an individual learner or with small groups of children. Such small-scale teaching experiments in which learning is guided and stimulated almost individually,are essential for theory-building because they make it possible to observe and record the effects of all sorts of interventions on the course of the learning process with a high degree of precision. This methodological proposal meets Resnick's (1980) comments during the 1980 AERA-meeting in Boston on a previous classroom teaching experiment (De Corte & Verschaffel, 1980).

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OEDIPE DEVANT LE SPHINX

ETUDE DES BIAIS DE QUELQUES QUESTIONNEMENTS DE PSYCHOLOGIE APPLIQUEE

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Like the question asked by the Sphinx to OEdipus,most of the psychological tests intended to evaluate mathematical abilities are enigmas more than problems.
They are founded on two types of difficulties:to find what is the "good" property among all properties involved,to take off all the parasites surrounding a question which becomes then a trivial one.All these operations are up the stream of the mathematical behavior that they are intended to study.

> "Ce n'était pas au premier venu d'expliquer l'énigme,
> Il y fallait de la divination." -OEdipe -Roi(Sophocle)
> "Me voici,moi qui eus accès au chant de la
> belle et céleste victoire,
> En déchiffrant l'indéchiffrable énigme de la Vierge"
> Les Phéniciennes(Euripide)

Pour pouvoir poursuivre sa route vers Thèbes et son destin,OEdipe eut à affronter le sphinx et son énigme.On sait que la question à laquelle étaient soumis les passants était"Quel est l'animal,qui le matin marche à quatre pattes,à deux à midi,à trois le soir?" ."Chaque jour les Thébains se réunissaient en une assemblée uniquement consacrée à résoudre le cruel problème. Après chacun de ces congrès infructueux,le sphinx dévorait une victime sur la montagne.Enfin OEdipe vint qui trouva la solution".
Il sût répondre "l'homme" et,depuis lors,chacun d'entre nous a rencontré cette question légendaire,l'a cherchée,le plus souvent sans succès;puis, apprenant la réponse d'OEdipe,l'a trouvée évidente! La victoire d'OEdipe prouve-t-elle que lui seul savait que l'homme marche successivement à quatre pattes,puis sur ses deux jambes,puis aidé d'un bâton?Certes non;tous ceux qui s'étonnent de ne pas avoir trouvé la solution savaient bien cela.Il ne s'agit donc pas d'un"contrôle de connaissances"(pour parler en termes modernes) On dira qu'il s'agit d'un"contrôle d'aptitude"ou d'un "test d'intelligence". Je ne rappellerai pas les sophismes bien connus sur le fait que le mot "intelligence "utilisé ici n'est pas défini.Cherchons quelle aptitude peut être en jeu ici:certainement pas l'aptitude logique à faire une déduction mais plutôt l'aptitude à voir qu'une question peut en cacher une autre ,Le mot"énigme" signifiait originellement(το αἴνιγμα)"définition ambigue".Le Sphinx utilise une expression imagée en parlant de matin,midi et soir,il sait que le passant prendra ces expressions au pied de la lettre(et de même pour les pattes).L'aptitude à passer d'un niveau de langage à un autre est certainement

celle qui est le plus impliquée dans la résolution des énigmes et...de nombreux tests psychologiques dont nous allons parler.Il s'agira des tests,cependant,qui sont sensés évaluer des aptitudes *mathématiques*.Certes la"pensée latérale",mise en lumière par DE BONO(1969-1972)joue un grand rôle dans la recherche mathématique mais nous verrons sur des exemples qu'elle intervient ici d'une tout autre manière et que ces tests sont en fait des énigmes et non pas des"problèmes mathématiques".

I - DEVINER QUELLE EST LA"BONNE"PROPRIETE -

Lorsque le Sphinx parle de matin,midi et soir,doit-on comprendre qu'il s'agit d'une même journée et d'éclairages différents?Doit-on comprendre qu'il s'agit de métamorphoses d'un être selon les saisons?Un enfant moderne ne pensera-t-il pas aux divers aspects que peut présenter une même espèce d'animaux placés à des longitudes différentes?...La principale difficulté de la question est de *deviner* de quelle propriété le Sphinx veut parler,en choisissant parmi toutes les propriétés que l'on peut trouver aux éléments en jeu. Dans la plupart des questions de psychologie,il y a une *infinité* de propriétés qui peuvent être également envisagées de manière tout aussi légitime scientifiquement (c'est donc une situation totalement différente de celle d'un choix heuristique).

J'ai déjà parlé à la rencontre IGPME de Warwick (J.ADDA-1979) des questions en "etc" sous forme de suites à continuer:qu'il s'agisse de suites de nombres ou de figures(cf.par ex.les tests de SPEARMAN et de BONNARDEL),donner la "bonne réponse",c'est choisir parmi l'infinité d'extrapolations possibles celle qui correspond à un"prolongement canonique" de celle ,parmi toutes les propriétés communes aux éléments proposés,qui a été choisie par le questionneur et,puisque le test a été en général sérieusement étalonné,qui est choisie par la *majorité* des individus correspondant à la *norme*.Il s'agit donc là d'un critère indéniablement anti-mathématique.On peut s'inquiéter de voir de telles questions apparaître depuis peu dans des manuels de mathématiques comme si, par un renversement pervers,l'enseignement des mathématiques devait constituer un apprentissage pour la résolution des tests qui avaient ,initialement,été élaborés avec l'intention de détecter des aptitudes mathématiques.Les mathématiciens ayant largement proclamé que l'on devait ,au contraire,développer les possibilités d'imaginer des fonctions possédant des propriétés complexes; on aurait pu croire que cela conduirait les psychologues à renoncer à ces tests; curieusement,c'est l'effet inverse qui s'est produit:certains enseignants font désormais une formation aux tests plutôt qu'une formation mathématique(4).On trouve même,dans un manuel pour classes de CPA,l'exercice suivant:

▪ **3. Calcul mental :**
Complète les suites :
2 4 6 8 . . . . .
7 14 21 28 . . . . .
. . . 25 30 . . . .
. . . . 34 40 . . .
. . . . . . 80 . 100

On en arrive là à faire compléter des suites dont on ne donne que deux termes!C'est,bien sûr,parce qu'il y a une suite de suites et que la "bonne" propriété des deux premières est d'être des suites arithmétiques,alors toutes les autres doivent nécessairement être aussi des suites arithmétiques!

Les nombreux tests dits de "similitude" ou d"analogie" ou ceux du type "cherches l'intrus"sont basés sur cette même démarche:deviner la "bonne" propriété" et en trouver une autre,même astucieuse ,n'est pas souvent payant:

ex:Dans le WISC,à la question "similitudes-pour les sujets de 8 ans et plus non suspects de retard mental",pour "49 ;121" il est ordonné de noter 0 la réponse que je trouve excellente"la somme des chiffres est égale à 4(preuve par 9)";les "bonnes" réponses ,notées 2points étant"ce sont des carrés parfaits; ont pour racines des nombres impairs ou des nombres premiers"(il est également prévu de donner 1 point pour "ce sont des nombres impairs").

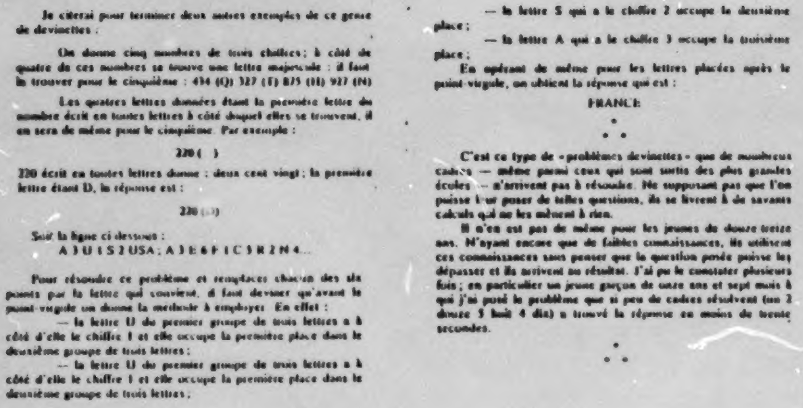
A ce sujet,le test dit"Test de raisonnement"me parait particulièrement interessant à analyser,d'autant qu'il est"très fréquemment utilisé par les cabinets de sélection"(citation de GOBET 1976).Il comporte 40 suites à continuer et les critères permettant de déterminer les "bonnes "propriétés y sont de nature très différente selon les suites;le "raisonnement"en jeu consiste surtout à penser à changer de niveau linguistique et cela différemment selon les cas.Dans son ouvrage "Les tests démystifiés "G.GOBET(1976) a bien caractérisé les divers types de critères:

ex:"un 2 douze 5 huit4 dix . "

dix ayant trois lettres ,3 est la réponse"...

... "Extrêmement rares sont les personnes adultes qui trouvent la réponse exacte de cet exercice dans le temps alloué par le test.Un seul auditeur , jusqu'à ce jour ,m'a déclaré être parvenu très rapidement au résultat en utilisant les deux procédés indiqués.Il avait travaillé au chiffre dans l'armée!"

Autre cas:une alternance de huit nombres écrits en chiffres romains avec sept nombres écrits en chiffres arabes;ilfaut,pour remplacer le point placé après le dernier nombre en chiffres romains,avoir remarqué que chaque écriture en chiffres romains est suivie du nombre de barres qui la composent!



Je citerai pour terminer deux autres exemples de ce genre de devinettes :

On donne cinq nombres de trois chiffres ; à côté de quatre de ces nombres se trouve une lettre majuscule : il faut la trouver pour le cinquième : 434 (Q) 327 (T) 875 (H) 927 (N)

Les quatres lettres données étant la première lettre du nombre écrit en toutes lettres à côté duquel elles se trouvent, il en sera de même pour le cinquième. Par exemple :

220 ( )

220 écrit en toutes lettres donne : deux cent vingt ; la première lettre étant D, la réponse est :

220 (D)

Soit la ligne ci-dessous :

A 3 U 1 S 2 USA ; A 3 E 6 F 1 C 5 R 2 N 4...

Pour résoudre ce problème et remplacer chacun des six points par la lettre qui convient, il faut deviner qu'avant le point-virgule on donne la méthode à employer. En effet :

— la lettre U du premier groupe de trois lettres a à côté d'elle le chiffre 1 et elle occupe la première place dans le deuxième groupe de trois lettres ;

— la lettre U du premier groupe de trois lettres a à côté d'elle le chiffre 1 et elle occupe la première place dans le deuxième groupe de trois lettres ;

— la lettre S qui a le chiffre 2 occupe la deuxième place ;

— la lettre A qui a le chiffre 3 occupe la troisième place ;

En opérant de même pour les lettres placées après le point-virgule, on obtient la réponse qui est :

FRANCE

⁂

C'est ce type de « problèmes devinettes » que de nombreux cadres — même parmi ceux qui sont sortis des plus grandes écoles — n'arrivent pas à résoudre. Ne supposant pas que l'on puisse leur poser de telles questions, ils se livrent à de savants calculs qui ne les mènent à rien.

Il n'en est pas de même pour les jeunes de douze-treize ans. N'ayant encore que de faibles connaissances, ils utilisent ces connaissances sans penser que la question posée puisse les dépasser et ils arrivent au résultat. J'ai pu le constater plusieurs fois ; en particulier un jeune garçon de onze ans et sept mois à qui j'ai posé le problème que si peu de cadres résolvent (un 2 douze 5 huit 4 dix) a trouvé la réponse en moins de trente secondes.

⁂

(Extrait de GORET (1976))

Notons surtout ,puisqu'il est sensé s'agir de "raisonnement" que la pensée latérale utilisée en mathématiques ne consiste pas à chercher des homonymes correspondant à une même désignation mais à chercher des procédés heuristiques - des modes de résolution- différents,d'un problème.Les exercices mathématiques comportant différents niveaux de langage,comme ceux portant sur la numération par exemple,sont relativement exceptionnels et souvent dus à de récentes traditions purement scolaires et fort artificielles ,et l'on sait qu'ils sont rarement réussis par les élèves (cf J.ADDA-1975).C'est,par contre,là,la source de la plupart des énigmes:cf.dans le Dictionnaire encyclopédique Quillet,la citation suivante ,dans la rubrique de définition du mot "énigme":

"Voici une énigme dont la réponse est"oiseau" et qui est due à VOLTAIRE:
Cinq voyelles,une consonne,
En français composent mon nom,
Et je porte sur ma personne
De quoi l'écrire sans crayon."

II-DECOUVRIR LE QUESTIONNEMENT DERRIERE SON HABILLAGE.

L'"habillage" d'un problème peut le camoufler(nous avons vu que le questionnement du Sphinx ne recouvrait aucune difficulté réelle qui aurait subsisté si la question avait été posée directement) et même induire des réponses fausses.

Souvent,c'est simplement la progression des questions d'une batterie qui constitue le piège en induisant de traiter la question n par le même procédé que la question n-1 (voir exemples dans J.ADDA 1976).

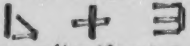
Les"épreuves d'arithmétique" du WISC comportent deux exercices avec un texte très embrouillé;l'enfant doit lire à haute voix avant de donner sa réponse au bout de 120".La difficulté de ces exercices est beaucoup plus de lecture que d'arithmétique.Je tiens à témoigner que je me suis moi-même trompée[1] et que, attentive au calcul,j'ai confondu les personnages et calculé ce que possède M.Lenoir et non ce que possède M.Dupont!

Le "Test de compréhension de concepts numériques" a pour but explicite "de situer le niveau atteint par l'enfant(entre 10et 13 ans)dans ce qu'on appelle communément "la comprehension du sens des opérations"".Sa première partie comprend 9exercices formés chacun d'une petite histoire suivie d'une question conduisant à un calcul et la réponse doit être donnée par une sélection dans un "choix multiple".Curieusement,les nombres qui sont proposés comme réponses sont écrits sous forme d'opérations et les questions ne portent pas directement sur les nombres que l'on veut f[illegible] trouver mais ce sont des questions au second degré commençant chacune par"comment fais-tu pour trouver..". Ainsi,lorsque l'histoire induit le résultat"10-2",la réponse "10-2"obtient 2 points mais la réponse "2+8",qui peut bien correspondre en effet à ce que certains "font pour trouver...",obtient 1 point!Il me semble vicieux de faire ainsi choisir entre un niveau mathématique et un niveau apparemment métamathématique,ou plutôt de confondre une réponse et une métaréponse.Notons que les autres réponses offertes sont,elles,facilement rejetables mais que l'enfant est certainement,par la forme des réponses proposées,induit à penser que la solution est un calcul et non un nombre.Une telle pratique risque de conduire à croire que 10-2 est très différent de 2×4 et presque égal à 2+8 !

La deuxième partie du même test est composée de 6 exercices:pour chacun d'eux on propose un calcul (à une opération)et on demande de rédiger une histoire qui y conduirait.La lecture du manuel de notation est succulente;on lit par exemple:"toutes réponses du type"J'ai une soustraction 64-16 et je trouve 48"ou "Un élève est convoqué au tableau:il a à faire une division 8:4.Combien trouve-t-il?."..seront cotées 0 car elles ne sont pas des illustrations valables de la compréhension conceptuelle des opérations"

Les "épreuves cliniques" de la "Batterie UDN 80" sont longuement analysées dans l'ouvrage de C.MELJAC(1979).On est stupéfait de voir que l'auteur espère étudier le dénombrement"spontané" avec des situations aussi artificielles. Par exemple ,dans l'épreuve dite"des poupées",on dispose des poupées et des robes sur une table puis on énonce la consigne:"Ouvre les yeux,regarde ces poupées;elles ont très froid,tu vas aller chercher les robes pour les habiller; Mais attention:elles veulent toutes s'habiller en même temps;il ne faut pas apporter de robes en trop;va chercher juste ce qu'il faut de robes".
Non seulement,il ne s'agit pas du tout d'une démarche spontanée mais ,de plus la consigne semble vraiment énoncée par un Sphinx:son décodage est autrement plus difficile que la résolution du problème numérique sous-jacent.

Enfin,on retrouve dans ces épreuves que l'auteur affirme d'ailleurs largement inspirées par PIAGET,le biais linguistique caractéristique de l'Ecole de Genève:l'utilisation d'un certain jargon qui ne correspond au langage d'aucun enfant réel ni en Suisse ni en France(cf J.ADDA 1980).
Exemple:"Tu vois cette poupée doit faire ce chemin(A) et cette autre,l'autre chemin(B).Est-ce que ce sera le même long chemin,est-ce qu'elles seront au bout du chemin fatiguées toutes lesdeux pareilles?".Le pauvre OEdipe aura bien du mal à comprendre que le Sphinx lui demande là tout simplement de comparer les longueurs des baguettes A et B.Les protocoles rapportés montrent bien d'ailleurs que ces questions sont vécues comme des énigmes.C'est par exemple le cas d'enfants "à qui il était demandé de calculer 6+3(avecliberté de se servir d'un matériel comme ils le voulaient)"et qui ont disposé les bûchettes de la manière suivante :



Nous avons vu que devant ces épreuves prétendûment "mathématiques",l'enfant interrogé rencontre comme principale difficulté celle de rechercher oùest le problème sous-jacent.Or,lorsqu'il l'a trouvé,il s'aperçoit,tel OEdipe,qu'il n'y a plus de problème du tout.Aucune réflexion mathématique n'intervient(on pourrait comparer,par opposition,ces tests ,par exemple ,à des exercices de type Olympiades ou même à certaines récréations mathématiques).Réussir ou échouer à de telles énigmes ne peut rien indiquer sur une quelgonque aptitude mathématique,il s'agit seulement d'une aptitude à passer des tests.

Notes
1-On dira peut-être que dans l'objectif de préparer leurs élèves à trouver un emploi,ils ont raison de les préparer à répondre aux épreuves de sélection des entreprises plutôt qu'à obtenir des diplômes universitaires!
2-Rappelons qu'il s'agit d'untest"pour les sujets de 8 ans et plus ,non suspects de retard mental".

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D

ENSEIGNEMENT SUPERIEUR
ATTITUDE ET ANXIETE ; BILINGUISME
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UNIVERSITY TEACHING
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TEACHER TRAINING.

THE MUTUAL RELATIONSHIP BETWEEN HIGHER MATHEMATICS AND A COMPLETE COGNITIVE THEORY FOR MATHEMATICAL EDUCATION

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Il existe une relation entre les mathématiques supérieures et la théorie cognitive qui devrait leur être d'un mutuel profit. La théorie cognitive sera enrichie si elle tient compte des exemples divers de la pensée mathématique, et inversement la théorie cognitive qui peut adopter ces modes de pensée peut contribuer à la compréhension des mathématiques.

Ainsi, la recherche dans ces domaines de la pensée peut être profitable de nombreuses façons:
1. Pour l'apprentissage des mathématiques au niveau des années terminales et universitaires.
2. Pour le développement d'une théorie cognitive plus complète de l'enseignement des mathématiques.
3. Pour la compréhension de l'aspect cognitif des mathématiques et de l'histoire des mathématiques, du processus créateur de la recherche et de l'attitude des mathématicians de métier envers leur sujet.

On considérera la recherche récente qui à étudié la différence entre les définitions mathématiques formelles et les significations personelles données aux concepts par les individus. Cette recherche a révélé des différences frappantes entre la théorie formelle et la perception qu'en ont les étudiants et mathématiciens professionels. Même si l'on enseigne des définitions formelles, l'imagerie conceptuelle de l'étudiant dépendra des expériences de l'individu et pourra être très différente de la théorie formelle.

La recherche met en valeur la question centrale de la "signification" en mathématiques et suggère qu'une théorie générale et adaptée à l'enseignement des mathématiques devra se fonder sur une acquisition "significative" de la connaissance, c'est à dire reliant la croissance des structures cognitives chez l'individu aux mathématiques à étudier et aux processus de pensée à développer.

INTRODUCTION

The psychology of mathematical education to date has been mainly concerned with the learning processes of children and the methods necessary to educate them in current mathematical theories. There have been far fewer studies of cognitive development at higher levels, with implications that cut two ways. On the one hand, the lack of knowledge of cognitive processes at more advanced stages of education can lead to weaknesses in the teaching of mathematics at college and university. On the other hand, the lack of study at this level has severely hampered the development of a complete cognitive theory of mathematical education by excluding the rich and varied examples of more sophisticated mathematical thinking. This lack of understanding of higher mathematical thinking has another serious implication: because the thinking processes of professional mathematicians are not well understood, this impairs our understanding of the nature of mathematics itself.

THE LEARNING OF MATHEMATICS AT COLLEGE AND UNIVERSITY

The most immediate application of cognitive studies at higher level is to provide a framework for the reassesment of teaching and learning of mathematics at college and university. My own work has concentrated mainly on the study of calculus and analysis: infinite processes, the concept of infinity, limits, continuity, differentiation, integration, the nature of number systems, use of infinitesimals, understanding of proofs, and so on.

A key idea that has helped in these studies is the distinction between a *concept definition*, which is the form of words used to describe a concept, and the *concept image*, which is the cognitive structure in the mind of an individual that is related to the concept (Vinner & Hershkowitz 1980, Tall & Vinner 1981). The concept image is more than a mental picture, for instance it is partially generated by the related processes experienced by the individual.

Suitably worded questionnaires have revealed the diverse nature of students' concept images in mathematics (see, for example, Schwarzenberger & Tall 1978, Cornu 1980, Tall & Vinner 1981). Mathematical terms like "function", "limit", "tends to", "continuous", and so on, all evoke a variety of concept images and the images evoked in a single individual can vary with the context. (Relevant examples will be discussed in Grenoble, but are omitted here because of limited space.)

The notion of concept image is useful for describing the development of understanding of axiomatic theories. For example, an answer to the question "what is a mathematical group?" might be to list the group axioms. But this is just the concept definition. To each individual the notion of a group is more than that: he has his own concept image (possibly empty) of the group concept developed through experience of manipulating the theory. This experience leads to a "feeling" for the concept generated by sensory input reacting with the concept image in his cognitive structure. In particular, each individual's intuition for a concept is a direct result of his own concept image.

The development of concept images may be usefully encouraged in the first place by presenting the individual with generic processes and generic examples: these are specific cases from which the individual can abstract the general theory. The technique is common in education at all levels, be it the interpretation of the specific statement $2+3=3+2$ as a generic example of the commutative law or the generic method of solving any given set of linear equations through a few well-known examples. Formally such examples play a redundant role in higher mathematics: and individual case never proves a general theorem. But in cognitive terms their use may be crucial because abstraction from generic examples seems to be an essential way in which human beings form concepts.

Investigations into conceptual imagery can lead to new strategies for teaching, by providing students with experiences that help in the creation of a concept image that is consistent with, and supportive of, the formal structure of mathematics. These experiences may themselves be formally unnecessary.

In analysis, for example, there is a school of thought which excludes the use of pictures because they are thought to give false intuitions. On the assumption mentioned above, that intuition is a direct result of concept image, it follows that true intuitions are more likely to come from a suitably developed concept image. By suitably formulating concept definitions, pictorial ideas may be used with great profit. As an illustration, one may define a function $f:D \to \mathbb{R}$ (from a subset D of the real numbers $\mathbb{R}$) to be *pictorially continuous* if over any closed interval $[a,b]$ in D,

$$\text{given } e > 0, \text{ there exists } d > 0 \text{ such that for } x,y \in [a,b],$$
$$|x-y| < d \quad \text{implies} \quad |f(x)-f(y)| < e.$$

It is easy to show that, given a pencil that draws a line of given thickness, the graph of a pictorially continuous function can be drawn to any specified scale over a closed interval $[a,b]$ in its domain without the pencil leaving the paper. What actually happens is that the graph lies *inside* the pencil line.

It is also easy to show that if f is differentiable at some point x_0, then given a piece of paper of any specified width and a pencil which draws a line of specified thickness, there is a small interval containing x_0 such that the graph over this interval scaled up to the width of the paper can be drawn inside a *straight* pencil line. This process can be exemplified using high resolution graphics on a computer, giving valuable cognitive support.

Based on these ideas it is easy to give students a recursive method of drawing an everywhere continuous, nowhere differentiable function. By physically drawing the successive approximations they may gain a psychomotor feeling for the properties of the function and by using the concept definitions and properties just mentioned these intuitions may be translated directly into a formal proof. (See Tall 1981.)

By reformulating mathematics, taking into account student's concept imagery, the theory may be enriched and made meaningful to a wider range of students.

A MEANINGFUL THEORY OF MATHEMATICAL THINKING

The study of mathematical thinking at higher levels demands an appropriate cognitive framework. In my own investigations, behaviourist theories which refuse to speculate on the nature of the thinking process have proved to be of little practical value. An extension of Piaget's theory of stages to higher levels also seems inappropriate. It is my belief that the best kind of overall theory of cognitive development is one which relates the developing cognitive structure of the individual to the conceptual framework that he either creates or is expected to master. Two useful existing theories which satisfy these criteria are those of Ausubel et al. 1978 and Skemp 1979; they both apply to *all* individuals at *all* ages.

In a meaningful learning theory, the individual's concept image of the mathematics he is expected to master is of paramount importance. The cognitive development is likely to pass through transition phases where new information causes a restructuring of the concept image; this may involve a period of conflict before the resolution leads to a new stage of thinking, as observed by Piaget. But the theory would suggest not a small number of Piagetian stages, but many transistions in many conceptual areas throughout life. It is the study of such transitions and how they may be effected which I believe to be a matter of central importance in a cognitive theory of mathematical education.

THE NATURE OF MATHEMATICS

Given an adequate cognitive theory, the study of the processes of matnematical research may reveal insights into the nature of mathematics itself. A recent personal investigation (Tall 1980) confirmed the classical accounts (e.g. Hadamard 1945) that the activity is anything but logical, with the individual doing the research painfully putting together conceptual images from his cognitive structure, groping intuitively for new patterns (often inaccurately) long before they could be logically verified.

There is a subtle blend of choice and consequence in research: the mathematician chooses (or *invents*) his starting points, implicitly or explicitly (these may include his concept definitions, his axioms and, to a certain extent, his rules of procedure) but from then on there are logical consequences implicitly built into the system which he must *discover*.

Educationists would do well to note this balance of choice and consequence, invention and discovery, in mathematical theories. Many decisions in mathematical education have been based on *arbitrary* starting points, chosen by mathematicians for mathematical reasons, and such starting points may be inappropriate for cognitive development. For instance, Piaget's notion of conservation of number is implicitly built on Cantor and Frege's choice of one-one correspondences between sets for the starting point for the theory of cardinal number. The mathematical theory was never intended to take into account the cognitive development of the child, where repetitive processes of counting fit naturally into the human development of action schemata. A reappraisal of the theory of cardinal numbers, as in Freudenthal 1973 or Stewart & Tall 1979, shows that the emphasis on one-one correspondence at the expense of counting is unwarranted.

The mathematics educationist therefore needs a flexible view of mathematics, one which attempts to see it therough the eyes of the learner and reformulates the structure in a potentially meaningful way. In doing so, one cannot escape the need to know something of the higher realms of mathematics, so that it can be made the servant of the educational process rather than the master.

Thus the circle closes: a theory of cognitive development enhanced by studies in higher mathematics may be applied to understand and modify the higher mathematics itself.

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APPRENTISSAGE DE LA NOTION DE LIMITE : MODELES SPONTANES ET MODELES PROPRES .

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Within the mathematical activity, mathematical notions are not only used according to their formal definition, but also through mental representations, which may differ for different people . These "individual models" are elaborated from "spontaneous models" (models which pre-exist, before the learning of the mathematical notion, and which originate for example in daily experience) , interfering with the mathematical definition . In this paper, we study the spontaneous models and the elaboration of the individual models for the notion of limit among students . We notice that the notion of limit denotes very often a bound you cannot cross over, which can or cannot be approached . It is sometimes viewed as reachable, other times as unreachable . The term "tends to" is also used with quite distinct meanings, which do not always agree with correct mathematical usage . We relate this study to the historical evolution of the concept of limit .

MODELES SPONTANES ET MODELES PROPRES

L'activité mathématique ne se réduit pas à la mise bout à bout de propositions selon des règles logiques; un objet mathématique n'est pas mis en jeu uniquement d'après les axiomes ou les propriétés qui le caractérisent. Le mathématicien - professionnel...

ou élève - met en jeu à chaque instant des aspects très personnels de la notion mathématique qu'il manipule : il a présents à l'esprit des exemples particuliers, chaque notion déclenche en lui des images, et ce sont ces images qui font fonctionner bien souvent l'intuition. La représentation mentale que le mathématicien a d'une notion donne à cette notion son aspect dynamique, vivant . Elle permet au mathématicien de faire fonctionner la notion. Au contraire, la définition mathématique formelle est bien souvent figée ; elle resta bien sûr le recours et le garant permanent, et elle permet l'écriture et la communication. Mais à elle seule, elle ne suffit pas à déclencher l'activité mathématique . Il est intéressant d'étudier la formation de cette représentation mentale des notions mathématiques, et il est important de la prendre en compte dans l'enseignement (cf. Tall & Vinner, 1981) .

En ce qui concerne la notion de limite, il se trouve que bien avant d'en avoir commencé l'étude en classe, les élèves en ont déjà une idée, provenant de la vie quotidienne : le mot "limite" s'emploie en français courant, il a un certain nombre de sens, comme nous le verrons par la suite. Ces sens sont la plupart du temps différents du sens mathématique. L'élève à qui on va enseigner la notion de limite a donc déjà en lui ce que nous appelons des *modèles spontanés* . Là dessus, le professeur va donner la définition mathématique, assortie d'exemples et de théorèmes où l'objet mathematique intervient et fonctionne. Il serait illusoire de penser que la définition mathématique va effacer toutes les conceptions antérieures de l'élève, en prendre la place pour donner lieu a un *modèle mathématique* qui désormais sera seul à intervenir dans l'activité mathématique . Bien au contraire, cette définition va entrer en conflit avec les modèles spontanés. Il va se produire des mélanges, des adaptations, pour finalement aboutir chez l'élève à des modèles engendrés à la fois par les modèles spontanes et la définition mathématique : nous les appelons *modèles propres* . Sur une même notion, il peut y en avoir plusieurs chez un même élève. Ils peuvent être (et ils sont bien souvent) inexacts sur le plan mathématique . Mais, lorsque l'élève aura à résoudre des exercices ou des problèmes, il mettra en jeu ses modèles propres, et non la notion mathématique à l'état pur . Ainsi, la plupart des erreurs faites par les étudiants à propos de la notion de limite ne sont pas le fait uniquement du hasard ou de l'inattention, mais elles sont la conséquence logique de leurs modèles propres. L'étude des erreurs permet d'ailleurs de retrouver les modèles propres des étudiants (cf. les travaux de Aline ROBERT sur la convergence des suites). Par rapport à la notion mathématique, les modèles propres ne sont ni totalement faux, ni totalement justes. La plupart du temps, ils sont issus d'exemples, et ils peuvent donc être opérants sur certains exemples et sur certains types d'exercices. Les exercices qu'un étudiant rencontre lors de sa scolarité sont assez semblables les uns aux autres, et on voit souvent des étudiants réussir leurs études sans trop de difficultés avec des modèles propres mathématiquement faux, car ces modèles propres ont été suffisants pour le champ couvert par les exercices rencontrés. C'est en prenant des exercices inhabituels, allant à contresens des schémas classiques, qu'on arrive à repérer les inadéquations de certains modèles propres . Les modèles propres ont aussi un caractère évolutif : Au fur et à mesure qu'on les utilise, ils s'affinent, se précisent, se corrigent . Mais ils peuvent rester éloignés du modèle mathématique fort longtemps .

LA NOTION DE LIMITE

Nous avons cherché, au moyen de différents tests (cf. Cornu, 1980), à déceler la signification du mot "limite" et de l'expression "tend vers" chez des élèves, juste avant qu'ils reçoivent un enseignement sur la limite, en classe de seconde . Les mêmes tests ont été proposés ensuite à des étudiants de différents niveaux, de façon à voir l'évolution des réponses selon l'avancement des études. De ces tests, nous avons tiré un certain nombre de renseignements sur les modèles spontanés :

En ce qui concerne l'expression "tend vers", on observe d'abord qu'elle ne fait pas vraiment partie du vocabulaire usuel des élèves de seconde. Ils ont du mal à donner des exemples de phrases comportant cette expression. Bien souvent, l'expression "tendre vers" remplace "avoir tendance à"; elle peut ne pas contenir d'idée de variation effective : " Ce bleu tend vers le violet" , ou au contraire traduire une évolution : " Ce régime politique tend vers le socialisme" . Dans un contexte mathématique, on a pu distinguer quatre modèles dans l'esprit des élèves :

Modèle a : tend vers = se rapproche de (éventuellement en en restant éloigné). Ainsi, si une grandeur augmente par exemple de 1 à 3, on peut dire qu'elle tend vers 10 .

Modèle b : tend vers = se rapproche de ... jusqu'à l'atteindre . Par exemple, si x augmente de 1 à 3, alors 1+x tend vers 4 . Il peut y avoir évolution avec le temps : dès qu'on a atteint la valeur désignée, "çà ne tend plus" !

Modèle c : tend vers = se rapproche de ... sans jamais l'atteindre . Par exemple, 1/x tend vers 0 lorsque x tend vers l'infini .

Les modèles a, b, c contiennent la notion de variation : pour qu'une grandeur tende vers un nombre, elle doit varier. Une fonction constante ne peut pas tendre vers quelque chose .

Modèle d : tend vers = "a tendance à ressembler à" , "est voisin de" . Par exemple : 2,8 tend vers 3 .

Le mot "limite" est évidemment plus habituel dans le langage quotidien des élèves. Il désigne presque toujours quelque chose de statique, de fixe : limite géographique, limite à ne pas dépasser (morale ou reglementaire), borne que l'on s'interdit de franchir : "les limites de la condition humaine"..". Là apparait la notion de difficulté à atteindre la limite, et donc la notion de "se rapprocher indéfiniment". Parfois, la limite est ce qui sépare deux choses : la limite entre un champ de blé et un champ de maïs ; le nombre 0 est la limite entre le positif et le négatif . Mais le plus souvent, la limite est la fin : il n'y a rien de l'autre côté . Les modèles principaux sont les suivants :

Modèle α : Une limite est infranchissable, c'est une borne .

Modèle β : Le modèle qu'ont certains élèves coïncide avec la notion de borne supérieure ou de borne inférieure .

Modèle γ : La limite peut être atteinte .

Modèle δ : La limite est impossible à atteindre .

Le caractère infranchissable de la limite est prédominant . Celà aura des conséquences dans l'activité mathématique . On notera que pour beaucoup, la notion de limite ne contient aucune idée de variation, de mouvement, de rapprochement de cette limite

En général, "limite" et "tend vers" ne s'emploient pas dans le même contexte . La limite désigne quelque chose de précis, alors que l'on peut tendre vers quelque chose de plus vague . Un exemple : on dira que la suite

0,9   0,99   0,999   0,9999   ...

"a pour limite 1", ou "tend vers 0,9999..." . Ou encore, que la suite $1-1/n$ "a pour limite 1", mais que la suite n^2 "tend vers l'infini". Pour certains élèves, une suite illimitée n'a pas de limite ... puisqu'elle est illimitée . On constate que quelques étudiants réservent le mot "limite" aux suites dont la limite est atteinte, et emploient l'expression "tend vers" pour les suites dont la limite n'est pas atteinte.

Il est intéressant de constater que les modèles identifiés chez des élèves de seconde se retrouvent dans les modèles propres des étudiants de tous les niveaux . Les tests que nous avons fait passer nous ont montré que même chez des étudiants très avancés, la notion mathématique n'a pas pris la place des modèles spontanés . Leurs modèles propres sont extrêmement marqués par la conception initiale .

LA LIMITE ET SON HISTOIRE

Avant d'arriver à la notion de limite que nous connaissons tous maintenant, les mathématiciens ont eu beaucoup de mal à préciser cette notion. L'étude de l'histoire de la notion de limite permet de voir que la plupart des modèles que nous avons rencontrés chez des élèves ont existé et ont joué un rôle dans l'évolution de la notion de limite. On employait encore au siècle dernier le mot "limite" pour désigner les bornes d'un intervalle ; le débat pour savoir si la limite peut être atteinte ou non, si l'on peut se rapprocher indéfiniment d'un point sans le toucher, a été au coeur de la construction de l'analyse, en particulier du calcul différentiel, au XVIII° siècle . Les façons d'opérer des grands mathématiciens d'alors apportent un éclairage intéressant pour la compréhension des modèles qu'ont les étudiants aujourd'hui. L'évolution historique permet aussi de situer les réelles difficultés de la notion de limite, et de comprendre que la définition mathématique ne suffit pas à effacer toutes les difficultés .

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L'ACQUISITION DU CONCEPT DE CONVERGENCE DES SUITES NUMERIQUES DANS L'ENSEIGNEMENT SUPERIEUR.

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Our purpose in this article is to show how to acquire in College Studies the concept of the convergency of infinite sequences. We have elaborated a list of exercices and questions meant to allow the students who answer them a way to express their views of the convergency of infinite sequences. Among the 1370 papers collected in the 1979-80 school years, we have been studying the expressed patterns, procedures and errors, the relationships between them and how they will all be altered during the 4 college years. Although this subject is only thaught during the first year, noticeable changes are observed throughout the following years, both in reference to the expressed patterns and the procedures.

Notre objectif est de décrire l'acquisition du concept de convergence des suites numériques dans l'enseignement supérieur (français) - grâce à une étude synchronique de productions écrites en temps limité par des étudiants en cours de scolarité universitaire (à divers niveaux). Nous avions établi à cet effet un questionnaire comportant à la fois des exercices classiques sur les suites numériques et des questions destinées à faire exprimer aux étudiants interrogés les "représentations" qu'ils se font de la convergence des suites.
Nous avons mené une étude clinique et statistique (1) de 1370 copies d'étudiants de classes préparatoires aux grandes écoles, de premier cycle universitaire scientifique et de second cycle universitaire de mathématiques - toutes les copies ont été recueillies en 1979-80.
Nous nous sommes efforcés d'analyser les représentations (ou modèles) exprimées, les procédures utilisées au cours des exercices et les erreurs auxquelles elles peuvent donner lieu, la relation entre les modèles exprimés et les procédures, et la répartition des conduites observées dans les copies au fur et à mesure du déroulement des études universitaires.

Nous avons constaté une évolution notable des modèles exprimés, non sans rapport d'ailleurs avec ce que l'on peut observer historiquement au fur et à mesure de l'élaboration du concept jusqu'à sa forme actuelle (2). On remarque par exemple

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(1) En collaboration avec Madame J. Mac Aleese, statisticienne.

(2) Cf. le travail de Madame M.C. Bour.

l'apparition croissante de l'expression du modèle "mixte" caractérisé par une description à la fois "dynamique" (u_n (terme général d'une suite) se rapproche de sa limite) et "statique" (traduction de la définition de la convergence en ε et N). Corrélativement, les représentations monotones de la convergence ("les suites convergentes sont les suites monotones bornées") que l'on rencontre en début de scolarité universitaire disparaissent des copies.
On note dans tous les exercices une augmentation simultanée de l'expression des modèles mixtes ou statiques et des procédures conduisant à la réussite des exercices; le modèle dynamique, quant à lui, s'avère en général non discriminant par rapport aux performances. Toutefois la relation entre le modèle exprimé dans une copie et les procédures qui sont développées au cours des exercices est variable suivant les tâches considérées et suivant le niveau, ce qui nécessite une étude précise.

L'étude détaillée des solutions de chaque exercice par chaque étudiant a permis de dégager des types de procédures - "algébriques" (souvent erronées), "formelles" (basées sur l'exploitation de la définition), utilisant tel ou tel théorème sur les suites, etc...
Si la répartition des procédures utilisées s'avère variable (là encore) selon la tâche considérée, elle semble au contraire bien cohérente dans chaque copie et permet d'esquisser des types de comportement (conduites).
De plus, on peut noter, outre la relation avec les modèles evoquée ci-dessus, l'augmentation simultanée du nombre d'années d'études (1) et l'utilisation de procédures conduisant à la réussite, et, le cas échéant, parmi ces dernières l'augmentation de celles qui présentent des changements de stratégie en cours de démonstration.

Il s'avère enfin que les erreurs commises par les étudiants ne sont généralement pas fortuites. On trouve, par exemple, à l'origine des erreurs, des lacunes dans les connaissances antérieures, des émergences réductrices de certains modèles exprimés, l'application erronée de théorèmes, l'application de théorèmes erronés (théorèmes en acte), etc... Nous nous sommes intéressés plus particulièrement aux cas de "reconnaissances de formes" où seule la forme du théorème ou de la définition est prise en compte, ce qui peut amener à des conclusions dénuées de sens. Nous avons aussi remarqué que la production et donc l'utilisation de théorèmes en acte relèvent souvent de la méconnaissance de certains présupposés qui existent dans les énoncés des théorèmes corrects dont sont issus les "théorèmes" erronés. Par exemple, l'existence de la limite est un présupposé dans des théorèmes algébriques sur sa valeur - même si cette existence est supposée explicitement au début de l'énoncé, elle est présupposée au moment de la formulation algébrique du théorème.

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(1) Compté au moment du passage du questionnaire.

Dans toutes ces analyses, nous avons constaté des différences importantes entre les filières classes préparatoires et premier cycle universitaire. Il n'en reste pas moins que, même si on se limite aux filières les plus "efficaces" quant aux performances, il y a suffisamment d'évolution entre les conduites de première année et de dernière année d'enseignement supérieur pour qu'on puisse affirmer que, en ce qui concerne cette notion de convergence de suites (réputée acquise en première année) l'"acquisition" continue les années suivantes.

Pour terminer, nous voudrions insister sur la mise en évidence des différences entre les individus recevant un même enseignement et entre les divers enseignements.

Les premiers résultats obtenus vont nous permettre de continuer ce travail en affinant l'étude des rapports entre l'enseignement dispensé aux étudiants (1) et l'acquisition individuelle du concept.

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(1) Cf. le travail de Madame F. Boschet sur les manuels de premier cycle universitaire.

CONCEPTUAL DIFFICULTIES FOR FIRST YEAR UNIVERSITY STUDENTS IN THE ACQUISITION OF THE NOTION OF LIMIT OF A FUNCTION

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RESUME

Dans cette communication nous voulons attirer l'attention sur quelques difficultés rencontrées par des étudiants entrant en première année de l'université. Ces étudiants ont choisi une spécialisation en mathématique ou physique et ont reçu, au cours de l'enseignement secondaire une initiation à la théorie des fonctions. Ils ont été suivis de près pendant les trois premiers mois de leur séjour à l'université. Les difficultés rencontrées au début du cours d'analyse mathématique, en particulier avec la notion de limite permettent de constater l'existence, chez ces étudiants, d'un image concept de la notion de fonction qui ne correspond pas entièrement à la définition formelle d'après Dedekind et qui par conséquent entrave considérablement l'introduction de la définition (ε, δ) de la notion de limite. On constate que le concept image de fonction peut être situé à deux niveaux, au niveau I, le plus élémentaire, l'étudiant identifie une fonction à une courbe ou un graphe, au niveau II, plus évolué, il l'identifie à une formule. Il semble être très difficile de s'affranchir des limitations imposées par cette conception. On peut être même y ajouter un niveau III où, plus généralement, on conçoit qu'une notion mathématique soit convenablement introduite en assemblant judicieusement une chaîne de symboles, procédant par analogie avec une définition rencontrée antérieurement.

§ 1. INTRODUCTION

The paper concerns the difficulties involved in the acquisition of some concepts of mathematical analysis by 18-year old students, beginning their first year at the university. The sample consisted of a group of 52 students, all having choosen a specialisation in mathematics or physics, hence they may be considered as well disposed towards the subject and entered university with the reputation of being a good student of mathematics at the secundary instruction level. The following observations are the results of a series of tests which underwent this students in the first three months of their instruction and of a series of lectures, working-sessions and informal conversations with them.

During the two or three previous years the students' instruction included an introduction to the fundamental ideas of mathematical analysis, with emphasis on the techniques of the calculus (representation of function, calculus of derivatives and primitive functions, limits). The development of theoretical notions was not very elaborate.

§ 2. THE CONCEPT IMAGE OF A LIMIT

In a logical development of the theory of functions of real variables, the first fundamental notion met with is this of a limit. It is an occasion to test the real level of mathematical understanding and ability towards abstract thinking of the students. It is also a representative test, for the result are, later on, confirmed in introducing notions of convergence of sequences and series, topology and integrals (with the current confusion of integrals and primitive functions). All this topics show us the same slow evolution from a rather visual concept image towards the understanding of the very abstract idea.

The starting point in introducing limits is the intension to study the behaviour of a function in a neighbourhood of a point x_0, which is admitted by the students of being useful without further difficulties, on the basis of some striking examples such as

$$f(x) = \frac{1}{x}, \quad \text{or} \;..\; = \frac{\sin x}{x}, \quad \text{or} \;,.\; : \sin(\tfrac{1}{x}), \quad \text{etc.} \qquad (x_0 = 0)$$

Developing a correct mathematical intuition of the limit, the aim of the course is to arrive at a full understanding of the formal definition, in our case

$$\lim_{x \to x_0} f = L \quad \text{iff} \quad \forall \varepsilon > 0, \quad \delta > 0$$

such that $\forall x \in \mathrm{Dom}(f)$, if $0 < |x-x_0| < \delta$, then $|f(x)-L| < \varepsilon$.

It is not difficult to obtain a verbal formulation of the idea involved in the formal definition : "if x is close to x_0, then f(x) is close to the limit L" is a commonly accepted statement. But the trouble begins in turning this statement into mathematical terms : it seems to be rather difficult to understand thoroughly

- the meaning of "x approaches x_0";
- the interconnection of the role of ε and η;
- the role and the order of the quantifiers $\forall$ and $\exists$;
- the insignificance of the case x=a and the value f(a);
- the fact that x_0 has to be a closure point of the domain of the function.

A clooser look at this problems reveals that the use of graphical representations may help to overcome the inherent difficulty of passing from a visual image into a formal definition. Most students stick to the graph : there are no difficulties if the function is capable of being fully represented, with all particularities, on a sheet of paper. But if a convenient visual aid is lacking the troubles rises again : about 50 % of the students were incapable to comment the absence of a limit for $x \to 0$ of the function $f(x) = \sin(\frac{1}{x})$, some of them were even incapable of describing correctly the behaviour of this function. And things are still worse with more complicated functions such as

$$f(x) = x \sin(\frac{1}{x}) \quad \text{or} \quad f(x) = (x \sin \frac{1}{x})^{-1} .$$

Meanwhile this remark furnishes us an explanation of the fact that students don't recognize the common nature of the limit of a function and the limit of a sequence or series (It is a pity that sequences are seldom endowed in textbooks with a graphical reprensentation).

Of course, closely related to the idea of curve or graph is the idea of formula and this implies a further step towards abstraction : the identification of a function with a formula. This is strongly encouraged by the numerous exercices found in textbooks. If the function is described by a formula, the whole machinery of theorems and rules (de l'Hospital), makes the calculus of limits much easier. Some students become true experts in this field.

§ 3. INTERPRETATION

The behaviour of first year students versus the introduction of limits is a typical illustration of the formation of a concept image of a mathematical idea (Tall and Vinner, 1980). We may say that a primitive concept image (level I) consists in the visual and extremely intuitive assimilation of a function with a curve or graph. Because all students are familiar with coordinate methods, this concept image is easily replaced by another one (level II) where a formula is used as a substitute for a function. The mathematical insight of a lot of students doesn't evolve further on. They assimilate a function with a computer (= formula) which furnishes the values of $f(x)$ for variable x and which is linked with a plotter, producing the graph. In this circumstances it is extremely difficult to explain the true meaning of the (ε, δ)-definition, because the modern Dedekind-definition of a function is lacking and it is the latter "static" idea which confers the limit its full signification. Of course, the Dedekind-definition is introduced at an early stage in secondary schools in the context of set theory, but most teachers don't like it and abandon it joyfully beginning function theory. However, we have to admit that the visualisation offered by Venn-diagrams is unable to replace the use of graphs. Nevertheless, the old "dynamical" view of a function is at the origin of most complications in explaining the limit and it requires a lot of work to banish misconceptions. Even current terminology is not very suitable (e.g. x "approaching" x_0).

The (imaginary) self-sufficiency of a formula has so great a power that it extends to the formation of concept images at a level III, which exceedes the domain of limits, namely the concept of the self-consistency of a string of symbols. In a few words we can describe it as follows : if we assemble in a certain, wellchosen order a set of mathematical symbols (the order is often a copy some other previously established formula), this assemblage acquires by its own a meaning and we have to discover this meaning by one or another trick (often by analogy with something already well known). In this way, the stringent requirement in mathematics of establishing a non-contradictory and non-void definition is greatly bypassed.

Examples of such concept images are (1) the use of the ∞ symbol : $\infty + \infty$ and $\infty \times \infty$ have a meaning, but $0 \times \infty$ has not, "is not defined" (but why? and why is $0 \times \infty = 0$ in measure theory?); (2) in integral calculus : given a function f, you write the integral sign $\int$ before it, and dx after, resulting in the string $\int f\,dx$ and this is the integral of f (but what about integrability of f?); (3) in convergence theory of series : for a given countable set of numbers $(x_n;\ n \in \mathbb{N})$, you write $\sum_{n=0}^{\infty} x_n$ and by the very act of writing it down there must exist a number called the sum of the series (without reference to all elementary notions of convergence).

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ASPECTS OF DEFINITION IN MATHEMATICS

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Cet article concerne des aspects du rôle de définitions en mathématiques. Le discussion est divisé en quatre. La première partie concerne la metaphore, qui existe en chaque définition grâce au présence du verbe "être", en faisant l'identification entre les deux parties.Pour discuter les mérites d'une définition on doit comprendre quelques de ses consequences.Pour eux découvrir, on doit l'accepter.

La deuxième est au sujet de définition vis-à-vis des organisateurs en avance(conception d'Ausubel). Il s'agit d'un filtrepour des situations mathématiques par rapport à l'identification des examples et des non-examples. Il faut contrôler chaque definition suggerée contre les faits qu'on voudrait achever par cette définition. Une example de "longueur"est discutée et aussi les relations entre l'intuition et la définition.

La troisième partie traite l'intention en mathématiques et la rôle centrale qu'il joue en decisions de jugement et de valeur. La discussion se tourne sur une définition de Euclid et une contre-example de cette définition de Connelley fait en 1978. Est-ce qu'on peut parler d'une vraie définition(ou une vraie conception)en mathématiques? On trouve que tout le temps, les mathematiciens tournent les processus entre objects(par exemple les numèros,les sequences,les permutations) et puis on cherche une definition.

La dernière partie discute courtement la conception de Imre Lakatos de définition preuve-genérée et aussi une quotation d'Aristotle sur la relation entre une définition et une manque de preuve.Il y a aussi un appendice sur des choses en mathematiques dont on peut vraiment parler de comprendre.

In this paper I intend to discuss some points concerning the nature and use of definitions in mathematics. The examples cited will necessarily be brief and will be dealt with at greater length at Grenoble.First let me set this paper in a slightly broader context.Any treatment of mathematical understanding should contend with specific features of mathematics itself.As Putnam said "Taxonomy without theory is blind" (1975).Too narrow or age-limited a study of mathematics can miss certain aspects which have ramifications throughout mathematical education.Appendix 1 contains a preliminary list of mathematical terms for which I feel it makes sense to talk about understanding.

A second general point concerns the way we inquire about understanding.The question "Do you understand" tends to force a dichotomous yes/no answer and encourages the view that there are only two states of understanding,all or none.A far better question would be "Show me your understanding. Let me see the nature,quality and extent of it".For me,understanding is a far vaguer notion.My understanding of something waxes and wanes,it alters continually, fading in and out,but,as Larry Copes claims,"it deepens as the number of perspectives I have on it increases".Consider the many diferent ways of looking at angle or fraction.No one way is sufficient by itself.Below are comments on some aspects of definitions that I would like to discuss.

a) Metaphor:(see Pimm(1981a) for further remarks on this topic)
Definitions create identifications.The metaphoric quality arises initially from the unvarying use of the copula "to be" in that they assert an X is something.While this may have originally identified the literal meaning (e.g. multiplication is repeated addition),wih widening use the two ideas,formerly the same,become separated.Repeated addition,far from being a synonym, becomes merely one possible model for interpreting a statement about multiplication. Various metaphors are of use in understanding fractions.Fractions seen as operators provide a useful means for coming to grips with their multiplication (composition),but this is not helpful for making sense of their addition, so an alternative image is required.Neither view is the right way of looking at fractions.The judgement as to their relative worth depends on your intention, what you are trying to achieve(a point to which I shall return later).

At a higher mathematical level,consider the definition of the tangent space to a manifold in terms of derivations on it.Here the concept image differs widely from its definition.Why are we so willing to accept opaque definitions and work with them rather than question them?In part,to challenge a definition requires some understanding of its consequences-to discover some of the consequencesrequires working within the framework established by it.But isn't this just another instance of the indoctrination theory of education,just as we learn to perceive in part as the result of social conditioning(see Lakatos, 1976,p.17).There is a dearth of criticism in mathematics and many of the social characteristics of mathematics,those of choice among alternatives,negociability of meaning and concepts,exploration,committment,intention and so forth can be well illustrated by critical discussion of definitions(see Pimm,1981b).Such decisions are made about mathematical objects-what is the path we should follow among such conflicting criteria as intuitiveness,plausability,powerfulness,generalisability and simplicity?One guiding suggestion by Larry Copes is"given the chance to define something in a less-intuitive but more generalisable way,have the definition make sense,then lay the"groundwork" for later generalisation of the definition by proving the logical equivalence."

b) Definitions as advanced organisers:

Definitions allow us to focus attention on certain salient features of mathematical situations which often will become examples of a concept.An example has to be an example of something."This drawing was not in her mind an example of anything geometrical.It was not a consequence but a starting point.But after we have developed some theorems the drawings become examples."(Hawkins, 1980,p.44).Definitions therefore act as filters through which to view mathematical situations.We even can accord something the status of being an example or not.Is 1 a prime number or not?Well, it depends.It depends in part on whether or not we want it to be prime - for that decision will guide our choice and judgement of an appropriate definition of primality.We must have some understanding of a concept before we are able to judge whether or not a proposed definition for it is a good one."Any extensional criterion for a concept would have to be checked to make sure it gave the right results, otherwise the choice of the criterion would be arbitrary and unjustified." (Searle,1969,p.9).Consider,"The length of a parameterised plane curve (x(t), y(t)),where $a \leqslant t \leqslant b$, is defined to be $\int_a^b \sqrt{x'(t)^2+y'(t)^2}\,dt$ ".We check that the definition agrees with our known examples(e.g. length of a straight line, perimeter of a circle) and this encourages us to accept it as a valid characterisation of length. It also acts as a device for extension in that we can now calculate the lengths of curves we were previously unable to.It also suggests the possibility of non-rectifiable curves(since integrals sometimes fail to exist) and so acceptance of a definition may also entail a revision of our intuitions about a notion.The justification for calling the value of the above integral "length" however comes from the previous examples.

c) Intention in mathematics:

There is often a concealed proposition behind a definition,verifying that the definition does indeed do what we want it to do. The judgement of adequacy or otherwise depends on our intention, what we want the definition to do.Consider the following definition from Euclid IX,9. "Equal solid figures are those contained by similar planes equal in magnitude and multitude".From its use the required sense of equal is 'same volume'.David Fowler(to whom I am most grateful for this example)suggests a planar pseudo-Euclidean definition: equal polygons are those contained by lines equal in magnitude and multitude.If equal here were intended to capture 'same perimeter',then this is a satisfactory definition;if it were'same area',then it is an unsatisfactory one,for polygons are not determined by their sides(except for triangles)-we can flex them.I will illustrate Connelley's flexible polyhedron, an object which invalidates Euclid's definition and is thus a counter-example to a definition.Thus both definitions contain hidden theorems which turn out to be false,though in the case of the Euclidean one, it was only shown to be inappropriate in 1978.

What a definition is intended to capture is a primary basis for any judgement of its adequacy.

But would we ever say a definition(or concept) were the true one?Omar Khayyam in *Discussion of Difficulties in Euclid*,dealt with the parallel postulate and the definition of ratio in Euclid's Elements.He refers to two definitions, Famous Ratio(the celebrated,intricate Book V,5) and True Ratio,where the latter is based on the process of *anthyphairesis*-the so-called Euclidean algorithm.(In passing,Khayyam also makes the definition"like magnitudes... are those whose difference has a meaning".In other words, pairs of magnitudes to which the Euclidean algorithm can be applied-a completely functional and process-related definition).One reason for studying the history of mathematics is to discover instances of definitions(or ways of looking at certain situations) being superceded by others and the reasons for it. Fowler's work (1979, 1981) on pre-Euclidean mathematics is an attempt to reconstruct an older conception of ratio (based on the process of anthyphairesis) which he believes was erradicated by the more powerful,abstract proportion-theory methods to be found in Euclid Book V.He feels this reconstruction is necessary in order to make sense of much of *The Elements*,particularly Book II and Book X(the latter of which, in terms of the number of lines, comprises one third of the entire work).What were those Greek mathematicians trying to do which led to this mathematics?

True definitions would be of true concepts-objects in some unchanging,Platonic world.In the study of mathematics we turn processes into objects,for which we then seek definitions.We even reify mathematical activity itself, the posing and solving of problems, into mathematics, a body of knowledge.There are many examples of lower-level processes becoming the objects of study at higher levels.Initially we have the process of counting producing numbers as invariants of the construction.Differentiation moves from process to an element of a function space of operators.A permutation of a set becomes a thing, a sequence of numbers becomes a thing.However,in mathematics,one central criterion with regard to both concepts and constructions is usefulness, one of practicality and pragmatism.This implies that as our intention changes, our attention may change and with it this criterion of usefulness.We may see our mathematical objects differently or even our judgement as to what constitues an object of interest.We have changeable concepts in that their very existence seems to be dependent on our purposes. Fisher's (1966)article on the demise of invariant theory describes a ghost town in our Platonic world where no-one goes any more. The objects are no longer attended to and it is almost as if they were never there because we have forgotten them. Except of course they have influenced objects and ideas to which we do still attend which we hope to understand. But above all mathematics is a human mental activity and as such our concepts and their definitions change in accordance with our requirements.For example,is the change in the concept of group which can be seen throughout the nineteenth century a sequence of approximations to the true concept, or a reflection of the changing use to which it was being put(see Pimm 1979 for further details)? Mathematicians in the main have negociated a high degree of agreement about their constructs and the ability to agree over such characteristics of mental objects leads to the feeling of their having an independent existence.

Mathematics is above all a constructive activity. Our Platonic world is littered with such constructions, some unfinished(by our lights),some misshapen, some broken,some discarded,some currently being played with and explored.But it is not random discovery,it is building with an end in mind. We are shaping our Platonic world by our designs and we have the ability to pull down and rebuild.Why were certain definitions accepted over others which may have been simpler or more intuitive or despite their ontological shortcomings? In part because they were useful constructs which permitted the solution of problems. Pragmatism often rules in the world of mathematics also.As Hilbert said," In mathematics,as elsewhere,success is the supreme court to whose decisions everyone submits".

d) Proof-generated Definitions and Definition-generated Proofs:
No discussion of definitions would be complete without mention of Lakatos work in *Proofs and Refutations* on the tangled interrelationships between the notions of theorem statement, theorem proof and definition of concepts. There is no space here to do this notion justice(though I will make some remarks during my presentation). Let me end by quoting Aristotle on what at first sight seems to be a reverse instance of Lakatos' notion of proof-generated definition.
"It would appear that in mathematics too some things are difficult to prove owing to the want of a definition,for instancethat the line parallel to the side and cutting the plane figure(e.g.rectangle,D.P.) divides similarly the base and the area. But once the definition is stated,the said becomes immediately clear.For the areas and the bases have the same antanairesis;such is the definition of the same ratio."(Topics 158b29).

APPENDIX 1

Different aspects we can come to understand.

a) Concept:Whence does it arise,why is it useful,what is its use?Concept image.
b) Definition:What makes a definition a good one?Examinition of alternatives, definitions as advanced organisers,proof-generated definitions.

c) *Statement of a theorem*:Relating two or more concepts.

d) *Proof of a theorem*:Globally or line-by-line.What makes it a proof?Can you abstract a method(e.g. induction,contradiction,compactness).

e) *Problem*:Why is it a problem?What are the consequences of its solution? Heuristic and problem-posing as tools for further understanding(S.Brown, L.Burton,P.Halmos,G.Polya,M.Walter).

f) *Solution*:Why is it a solution?Justification,pragmatism(it works).Is it the only one,the best one?Grounds for acceptability,the social and negotiated qualities of the decision.

g) *Example/non-example/counterexample*:What is it an example *of*?Connelley's flexible sphere is a counterexample to a definition.Lakatosian theory-based taxonomy of local and global counterexamples and their uses.

g) *Axiom*:What makes it an axiom?Intuitive quality,basicness,inability to prove at this stage(e.g. $A=\pi r^2$).

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I am grateful for discussions on some of these ideas to Larry Copes and David Fowler.

THE PUPIL'S VIEW OF MATHEMATICS LEARNING

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Ce rapport représente les résultats de recherches examinant la réaction d'élèves de 14 ans à leurs expériences scolaires - bonnes et mauvaises. Ce rapport décoit en particulier les traits révélateurs dans les expériences d'enseignement de mathématiques. Il fut demandé a 84 élèves au cours d'entrevues semi-structurées de commenter les occasions lorsqu'ils eurent conscience d'être particulièrement bons ou particulièrement mauvais dans leurs études.
Une histoire consista en un événement "critique" réellement vécu par l'élève et relata ce qu'il ressentit à ce moment-là. La structure de l'entrevue utilisée et les moyens par lesquels les données qualitative furent analysées sont discutés ici.

INTRODUCTION AND METHODOLOGY

The research* described in this paper was concerned with an examination of how fourteen year old pupils perceived good and bad experiences associated with their learning in school, how and why they judged specific learning situations as good or bad and what they perceived to influence these judgments. An attempt was made to 'capture' these perceptions by asking pupils to tell stories about times during which they had felt particularly good or particularly bad when learning. The research also aimed to discover how frequently stories about mathematics, good or bad, might be told and to find out if these mathematics stories had any distinctive features in a comparison with stories about other areas.

In order to investigate the pupils' view of learning, 84 fourteen year old secondary school pupils gave descriptions of actual events (called stories) which they had experienced and which they felt had been particularly significant in their learning. This approach was used since

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*The research was conceived as an extension to a secondary school population of the Higher Education Learning Project (Physics) (Bliss and Ogborn 1977) which had studied how learning at university was viewed by Physics undergraduates.

- it allowed the pupils to talk about things that were real and meaningful to themselves;
- it enabled an analysis of learning situations to be made from the pupils' point of view, that is from their internal frame of reference;
- it allowed all factors perceived by the pupils as important to be brought into the analysis and did not require the restriction of attention to a limited set of pre-determined influences;
- it enabled the interviewer to penetrate the actual meaning of the pupils' descriptions if these were superficial, ambiguous or ill-defined.

A semi-structured interview was used to collect the stories.* A fixed schedule of questions was not appropriate in this interview since the pupil was free to describe any event, or sequence of events, that came to mind, but a systematic approach was adopted and an outline structure developed. Six stages of the interview were distinguished:-

- the informal introduction aimed at setting the pupil at ease and where the research is described in a chatty manner;
- the collection of pupil data;
- the formal introduction where the request for a 'story' of a critical incident is made;
- the elicitation of the concrete details of the event described;
- the elicitation of how the pupil had felt at the time of the event;
- the request for a further story.

The nature of the interview, with its probing style and reiteration of detail, was structured to make it difficult for a pupil to make up a story and not betray this through inconsistencies or contraditions. The interview was also specifically designed so as to put the pupil at ease and encourage him or her, in a non-directive and neutral manner, to talk openly.

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* The interview was based on the critical incident technique used by Herzberg (1967) in his studies of motivation to work and had also been used in the HELP(P) study.

ANALYSIS OF STORIES

After the collection of the stories, procedures of analysis were developed which aimed to be both flexible enough to catch the essence of the stories yet rigorous enough to allow comparisons to be made between stories. The taped protocol of each interview was transcribed verbatim and checked.

In order for a description to be accepted as a story for coding and analysis, the following three components had to be identifiable:-

The context or situation in which the story took place, called the situation;

The feeling expressed, called the feeling;

The factors which appeared to be associated with the feeling, called the reasons.

Any pupil descriptions which did not contain all of these three components were discarded, leaving a total of 281 stories available for analysis. These stories were then coded, that is summarised into a series of descriptive statements and fitted into a standard outline structure. The statements were firstly taken directly from the interview transcripts and then standardised. Two categorial schemes, one for feelings and one for reasons, were inductively developed for the statements, using an a posteriori approach to content analysis. A summary of the categorial schemes derived will be distributed at the conference.

DISCUSSION OF RESULTS

Descriptive research of this kind produces a complex array of findings, some of which are directly quantifiable and can be analysed for statistical significance, while others are more suggestive and insightful and though not widely generalisable offer useful insights.

Out of the total of the 281 stories collected in this research a significant proportion (114 stories, approximately 40%) was concerned with mathematics and this proportion did not merely reflect the time and emphasis given to the subject in the school curriculum. Nearly one-third of all good stories and one-half of all bad stories were, in fact, about mathematics learning. Out of the 114 mathematics stories collected, a significant proportion (over 63%) was bad. In all the other areas taken together the proportion of bad stories was less than half (44.3%).

It would be reasonable to assume that the frequency of recall of stories about mathematics is, to a certain extent, a reflection of the strength of reaction to learning experiences in the subject; that is pupils would be more likely to recall experiences to which they had reacted strongly than those which had a lesser effect on them. These findings, therefore, suggested that mathematics tends to provoke both strong and adverse reactions in fourteen year old pupils.

In a comparison of these mathematics stories with stories about other areas, it appeared that the major sources of satisfaction and dissatisfaction in the mathematics learning experiences were, in general, similar to those relating to other areas of learning in school; that is, satisfaction tended to be attributed to involvement or success in work and dissatisfaction more likely to be blamed on the teacher. However, within the sorts of reasons and feelings described in all the stories, some quite marked differences in emphasis were apparent in the mathematics stories.

Firstly, the stories showed quite clearly that pupils were much more concerned with their own role in relation to learning mathematics than learning other subjects. Pupils had strong ideas about what they were capable of doing and what they were capable of understanding in mathematics and their mathematical experiences were dominated by this focus on self and feelings about oneself. There was, however, diversity within the mathematics stories which suggested that pupils differed in the goals they set themselves with regard to mathematics. For example, some pupils liked being able to do their mathematics on their own and liked to know 'why' as well as 'how'; some pupils enjoyed challenge in the subject; some pupils were well satisfied if they could just grasp 'what to do' in order to reach a successful solution; a great many pupils were very concerned with the outcome of their work, its rightness or wrongness and the marks they received. Despite these individual differences of goal, however, the stories indicated that it was when a pupil failed to reach his or her particular goal, whatever it happened to be, that he or she began to doubt his or her ability. The following quotation from one of the interviews is given as an illustration of this tendency -

> "I just wanted to get something down on paper, that's all ... just be able to write down a few lines to show I'd at least tried and was not completely stupid. It was no good. I was just a failure ... I knew I would never be able to get anywhere with it, no matter how long I sat there ...".

Further investigation is needed in order to find out in more detail the types of goals to which pupils aspire in mathematics, how they come to choose these goals, and the consequences for them of failure to reach these goals.

The stories also showed that anxiety, feelings of inadequacy and feeling of shame were quite common features of bad experiences in learning mathematics. In addition, from some of these stories it is possible to speculate as to the type of situation which seemed to provoke or accentuate such feelings. For example there was some indication that pupils in mathematics were particularly fearful and resentful of teachers who seemed to impose additional demands on them. Pupils were appreciative of a secure, encouraging environment in their mathematics lessons and liked teachers to provide a structured logical progression in their work, with plenty of patient explanation, encouragement and friendliness. Pupils, therefore, seemed to want teachers to 'make it easy' or 'tell them the way', perhaps in order to relieve any tension they might feel in their mathematics learning.

CONCLUSIONS

The pupils' stories about mathematics learning in this research can be seen to highlight certain problems for the teacher and mathematics educator, firstly in terms of apparently conflicting expectations between pupils, and secondly in terms of pupil expectations which would appear to be at variance with good educational practice. For example, the stories indicated that pupils want security and structure in their mathematics, but the provision of too much structure would probably discourage creativity and exploration in the subject and mitigate against pupils taking any responsibility for their own work and progress. Pupils were extremely concerned with the outcome of their work, they wanted to 'do it', 'finish it' and 'get it right', but this very concern could mitigate against involvement in the subject itself.*

Pupils appeared to demand grades and assessment yet seemed to see these as 'information' as to their mathematical ability and therefore judged themselves highly if they did well in mathematics but found it difficult to rationalise any failure in the subject. This also seemed to lead them to associate such failure with feelings of inadequacy and anxiety. Pupils wanted to be given mathematics of an 'appropriate' standard but quickly lost confidence if teachers left them behind or put pressure on them. Pupils did not talk about what their mathematics was about, or how it may be used. They did not appear to see that the subject could be of any interest in itself but only as something to be done, something to be mastered, something with an existence of its own.

This research has left questions unanswered and avenues unexplored yet it has perhaps provided many insights and pointers for future work. It is perhaps appropriate and in keeping with the 'spirit' of the research to end with a quotation from one pupil after he had told his stories:-

> "I really enjoyed that, miss, you sitting there and listening to me - makes a change somehow, doesn't it?".

* The absence of this involvement, according to Lefcourt (1976) would, at least partly, explain why any anxiety in mathematics learning tends to be debilitating rather than facilitating.

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ASSESSMENT OF MATHEMATICS ANXIETY

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5

Abstract

Evaluation du "Mathematics Anxiety"

Ce qui suit donne un resumé des progrès fait sur le projet du present auteur et d'Eugène E. Levitt, directeur du departement de Psychiatrie, Indiana University.

L' anxiété induite par les mathematiques produit des sensations de tension et d'anxiété qui peuvent amener le sujet a éviter de prendre des cours importants de mathematiques et même à échouer a ces cours. Dans une étude conduite par Richard Suinn on a trouvé que plus d'un tiers d'étudiants, suivant un programme de rehabilitation psychologique, que l'anxiété induite par les mathematiques etait au centre de leur problemês.

Une performance faible en mathematiques limite serieusement la carrière de nombreux etudiants. Dans notre société technologique les mathematiques jouent un role important dans un nombre croissant de carrières.

La manifestation d' une forte anxiété chez le professeur de cours elementaires est un problême particulièrement grave. Ses étudiants (ages de 6 ans à 12 ans) sont souvent exposés a leur premiers cours et peuvent être fortement influencés par l'attitude de leur professeur.

Les buts de ce projet sont: 1) D'étudier la frequence de l'anxiété induite par les mathematiques dans certains cours; 2) D'étudier l'anxiété induite par les mathematiques en tant que fonction de l'anxiété induite par les examens; 3) D'étudier l'anxiété induite par les mathematiques en tant que fonction de certaines variables demographiques (age, genre, etc...); 4) D'etudier l'anxiété induite par les mathematiques en tant que fonction du succès dans certains cours mathematiques; 5) D'étudier l'influence de certains developpements en classe; 6) D'etudier l'effet des calculateurs sur l'anxiété induite par les mathematiques; 7) D'evaluer certaines methodes pour reduire l'anxiété.

Cinq questionnaires furent distribués a 1000 etudiants. Les resultats sont sous étude et seront donnés a la conference de Grenoble.

This is a progress report of an on-going research study by the author and co-investigator, Eugene E. Levitt, Director, Department of Psychiatry at Indiana University.

Mathematics is becoming an integral part of the preparation for an ever increasing list of careers. Our technological advancements are emphasizing the critical importance of mathematical knowledge in our developing society. However, many capable students entering college lack the high school prerequisites for entry level mathematics courses. These students are severely limited in their career choices. In a survey of a random sample of entering freshmen at the University of California at Berkeley in 1972, Lucy Sells reported that 57% of the males and 92% of the females lacked the mathematics prerequisites for any college-level calculus or statistics courses. Many of these students at our college take remedial mathematics courses. These courses have a success rate of only 50 or 60 percent. Sells identified mathematics as a "critical filter" in career selection (Sells, 1973).

This math avoidance and poor mathematics performance is currently being self-identified by many students as "math anxiety". The best seller book entitled "Overcoming Math Anxiety" by Sheila Tobias (Tobias, 1978a) popularized the term "math anxiety" as well as the idea that females were especially susceptable. One definition of math anxiety is "feeling of tension and anxiety that interfere with the manipulation of numbers and the solving of mathematical problems in a wide variety of ordinary life and academic situations" (Richardson & Suinn, 1972). Mathematics avoidance and poor mathematics performance are prevelant among women in the USA (Carnegie Commission on Higher Education, 1973; Betz, 1977).The reasons for male-female differences in mathematical performance are not clear. Social processes are given much credit for female math avoidance by John Ernest (1975, "Mathematics and Sex"). While Tobias (1978b) cites an unfortunate early experience with a particular mathematics teacher as one probable cause.

The debilitating effects of academic anxieties on student performance have long been recognized (Spielberger & Sarason, 1978). Educational and psychological researchers have been studying the effects of anxieties for many years. However, it is only recently that subject specific anxieties have been considered. Mathematics anxiety involves feelings of tension and anxiety that can result in avoidance of or failure in fundamental mathematics courses. In a study by Suinn it was reported that over one-third of the students responding to a behavior therapy program indicated that mathematics anxiety was at the center of their problems (Suinn, 1970). Of special concern are what appear to be clinical manifestations of strong mathematics anxiety among pre-service elementary teachers. The elementary teacher is usually the student's first formal mathematics teacher and thereby can have a profound effect on student attitudes. In a survey of 80 public colleges and universities representing the 50 states in the U.S.A., 24 baccalaureate degrees were ranked by mathematics requirements and correlated by degrees conferred by sex in 1974-1975 (Lavroff, 1980). Mathematics anxiety scores were a more powerful correlate to the choice of academic majors than sex. Female education majors reported the highest anxiety scores of any degree group. Lavroff reported that nearly half of these students reported that "fear of mathematics" had kept them from selecting the major they wanted.

The objectives of this study are: 1) To investigate the prevalence of mathematics anxiety of students in certain undergraduate mathematics courses; 2) To investigate the relationship of mathematics anxiety to general test anxiety; 3) To investigate the relationship of mathematics anxiety to certain demographic variables (age, sex, etc.); 4) To investigate the relationship of mathematics anxiety to achievement in mathematics courses; 5) To assess the causal impact of certain classroom factors on mathematics anxiety; 6) To investigate specifically the effects of hand calculators on mathematics anxiety; and 7) To evaluate certain methods for reducing mathematics anxiety.

We are concerned with the relationships among mathematics anxiety, general test anxiety, and anxiety proneness in general. We have collected data in a number of different mathematics courses such as beginning courses for business majors, for elementary education majors, and courses that are usually taken only by math majors. The following three anxiety scales were administered to a sample of 1000 students in these mathematics classes at the beginning of Fall 1980 semester:

1. The Mathematics Anxiety Rating Scale (Richardson, Frank C. & Suinn, Richard D., The MARS: Psychometric Data, Journal of Counseling Psychology, Vol. 19, 551-554, 1972). Data presented in this reference indicates that MARS is valid, reliable and has high internal consistency.
2. Test Anxiety Inventory (Consulting Psychologist Press). This instrument was developed recently by C. D. Spielberger of the University of South Florida. The TAI consists of 20 statements pertaining to how the individual generally feels; responses are recorded on a 4-point Likert scale.
3. State-Trait Anxiety Inventory (consulting Psychologist Press). The STAI scale is intended to assess individual differences in anxiety proneness. This scale consists of 20 statements pertaining to how the individual generally feels; responses are recorded on a 4-point Likert scale.

The responses to these three scales are being analyzed for consideration of the first three objectives listed above. The semester grades of students who responded to the three anxiety scales are also being compared with levels of anxiety to investigate the question of a relationship between mathematics anxiety and achievement in mathematics (Objective 4).

A Teacher-by-Subject survey was prepared and distributed to elementary teachers in the Indianapolis Public Schools (students age 6 years through 12 years). The survey asked the teachers to indicate their like-dislike for teaching the various subjects in the elementary school curriculum, e.g., writing, grammar, reading, mathematics, social studies and science. It also asked them to rate their preparation for teaching the various subjects. This survey was developed to consider the hypotheses: a) elementary teachers do not like mathematics, b) elementary teachers do not consider themselves well prepared to teach mathematics, and c) elementary teachers pass their dislike for mathematics on to their students.

Interviews with elementary education students who consider themselves to be math anxious gave rise to the thesis that mathematics students who are experiencing difficulties may have a generalized set of characteristics which they attriubte to mathematics teachers. If this is true, perhaps these characteristics will furnish clues to causal questions. To investigate this question a survey listing 17 personality traits and their antonyms was distributed to a sample of students. The results of this survey are "anxiously" awaited.

Analyses of the above data is currently in progress. Regretfully at this writing the results are not in reportable form. However, the results will be distributed in Grenoble and their implications for mathematics education will be discussed.

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MATHEMATICAL INVOLVEMENT - A SIGNIFICANT AFFECTIVE VARIABLE?

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Il y a un grand gouffre entre les recherches traditionelles au sujet des attitudes mathématiques des élèves et leur conduite en classe. Dans cet article l'auteur lie les recherches au sujet d'inquietude et celles de la conduite des professeurs, aux recherches récentes au sujet d' 'achievement motivation' et 'attribution theory'. Il propose une variable nouvelle 'mathematical involvement' qui peut nous aider avec les recherches observantes des attitudes en classe.

In this paper I want to bring together some ideas from different areas of research which all relate to the affective dimension of mathematics learning in the classroom. It is my impression that there is a gap between studies of the affective side of mathematics and the reality of the classroom, which I see as an example of a more general division between psychological research and classroom teaching (Bishop, 1980).

Traditionally research into pupils' attitudes to mathematics has involved some sort of self-reporting system relating to various scales. For example in the IEA study (Husen, 1967) five attitude scales were used, relating to mathematics as a process, the difficulties of learning mathematics, the place of mathematics in society, school and school learning, man and his environment. The approach was typically psychometric. No behavioural, individual observational evidence were used, only group written responses to written stimulii. There was much concern about the types of statements used, the reliability of the responses and the quantitative analyses of results.

There was however, little relationship between this type of research and what was happening in the classrooms. Occasionally affective measures have been used alongside achievement tests to evaluate teaching experiments, so at least there was an assumption that attitudes were modifiable though how modifiable is still unknown. The construct of 'attitude' seems to relate to rather deep-seated and persistent phenomena. Although the children arrive at a mathematics lesson with certain attitudes towards mathematics, and leave again with certain attitudes, we know very little about what occurred in between - how did their attitudes affect what happened in the lesson? How did what happened in the lesson affect their attitudes?

With an increasing concern for disadvantaged groups in our educational systems came a research focus on problems surrounding negative attitudes. 'Fear of mathematics' (mathophobia) and 'anxiety' studies have been undertaken with for example, slow learners, adults, girls, and black children in America, and their combined picture has shown us how frightening mathematics can be for some people. Moreover it is clear from these studies that the teaching situation is more to blame than the subject itself.

So, if we are to understand more about the mechanisms by which teachers modify pupils' affective responses then we must attempt to focus on the dynamics of the mathematics classroom. Fortunately observational research has developed considerably over the last decade and many interesting findings are emerging. For example, the process-product methodology of Good, Brophy, Grouws and others has thrown into question the teacher' use of praise. In terms of effective teaching and developing favourable attitudes to the subject one would expect teacher praise to have a significant effect. However Good and Grouws (1977) found that more effective teachers used comparatively little praise and criticism, and Good (1980) reports Brophy's findings that the use of praise seems to be "determined more by students' personal qualities or teachers' perceptions of students' needs for praise than by the quality of student conduct or achievement" (p.9).

Now it is the case that these studies use increases in pupils' achievement test scores as the criteria of effective teaching and we must be cautious about interpreting these findings in relation to pupils' affective development. Nevertheless research like this shows that "obvious" teaching strategies like the use of praise have a non-obvious relationship with pupil outcomes. Indeed Good and Grouws' (1977) research shows that other aspects of effective teaching could well have much more significance for the development of favourable attitudes - general clarity of instruction, a task-focussed environment, and higher achievement expectations, for example.

So what should be observed in observational research in order to understand better the affective side of mathematics learning in classrooms? In my view, in order to answer that question, we need to search for richer and more "surface" constructs than 'attitudes' and 'anxiety'. Such a search has taken me into the extensive literature concerning achievement motivation and attribution theory. The ideas of achievement motivation, and the 'need to achieve' have been around for quite a long time now (Atkinson, 1964) but have not really been taken up by the education profession. Perhaps this is due to difficulties of testing achievement motivation but also I think it is because this characterization of motivation does not offer much possibility for direct modification by the teacher. The pupil arrives in the classroom with certain needs and motivations, and Atkinson's ideas can certainly help us to appreciate why the pupil may or may not get any satisfaction from a particular mathematics lesson. But the idea of motivat*ing* the pupil, sounds rather like what football managers do to their teams, and does not have much educational appeal. Indeed Kelly (1955) would argue that 'motivation' itself is a redundant construct. Man is motivated, by definition; the question is towards what?

In the last decade, however, a theory has been developed which does suggest a way-in to the problems of helping children who appear to have either negative attitudes towards mathematics or motivations towards other things than mathematics in mathematics lessons. *Attribution theory* refers to the field of research concerned with how individuals explain the causes of events to themselves. Weiner (1974) has been the leader of a group trying to understand how individuals with different needs for achievement see the causes of their successes and failures. In particular they have looked at so-called 'internal' causes like ability, and effort, and 'external' causes, like task difficulty and luck. As well as the internal/external dimension there has been much interest in the 'stability' of these factors with ability and task difficulty being stable, whereas effort and luck are unstable.

Bar-Tal (1978) in an excellent summary article sets out the typical findings: after success, people feel most pride when they can attribute that success to either ability or effort (internal) and less pride when they think it was due to good luck or the ease of the task; failures attributed to lack of ability or effort result in shame, in contrast to attributions of failure to bad luck or task difficulty. If we then add in the achievement motivation differences it appears that pupils high in achievement-needs tend to attribute success to the external causes. More importantly, high N-ach pupils tend to attribute failure to lack of effort, which is unstable and therefore changeable, whereas low N-ach pupils attribute failure to lack of ability, which is stable and does *not* allow for the possibility of change.

What are now beginning to emerge in the literature are studies which show that by providing learners with appropriate feedback they can be encouraged to change their attribution patterns. For example, Dweck (1975) carried out a study with elementary school children who showed 'helpless' behaviour i.e.

giving-up in the face of failure. They were attributing their failure to lack of ability, but by giving them particular verbal feedback during the training sessions Dweck taught them to attribute failure to lack of effort. The results showed that the children continued in this and began to improve their performance. These findings have been replicated and more studies are appearing.

This has been a brief and therefore superficial overview of what is a highly complex area (see, for example, Covington et al., 1980, but we know enough from existing studies of teacher-pupil interaction to realise the significance and potential of teacher expectations and feedback following successful or unsuccessful learner performance. For example, we know (now) that praise following success is not necessarily the best thing to offer. Further challenges, or analysis of the reasons for that success may be more beneficial.

What we need now are studies which explore these ideas in the mathematics classroom, and in order to do this we need to choose,amongst other things, what to observe. I have decided to focus initially on what I call Mathematical Involvement - the engagement of a pupil in a mathematical task. It is a dependent variable - dependent on the teacher, the pupil and the context. It is also assumed that it has a controlling effect, in the sense that if the pupil is not mathematically involved then there is little chance of him producing successful performance.

It is a variable, in the sense that a pupil at any one time may or may not be mathematically involved. It will therefore be possible to use the critical incident technique which I have used previously in work on teacher decision-making (Bishop, 1976), to study the reasons offered by the participants for the discontinuities in involvement. Mathematical involvement relates to effort and task-difficulty (attribution theory), and also to other phenomena analysed in classroom observational research, like 'time-on-task' and 'engaged time'.

The important teacher behaviours which will be studied will relate to initiating mathematical involvement by the creation of tasks appropriate to the pupils and sustaining it , by encouragement, feedback and by the control of extra-individual aspects (peers, physical constraints etc.).

The aim will be to explore the relationships between teacher behaviour, mathematical involvement and pupils' attributions, and hopefully to see whether teachers can be encouraged to modify and 'educate' those attributions.

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BILINGUALISM AND REASONING IN MATHEMATICS

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Dans cette communication, on essayera de présenter le contexte dans lequel s'est déroulée une étude sur des enfants bilingues qui apprennent les mathematiques dans une langue différente de leur langue maternelle dans des écoles anglaises.

L'énchantillon consiste de 203 enfants bilingues et de 167 enfants monolingues de langue anglaise, qui ont entre 11 et 14 ans. Les sujets venaient de 5 écoles anglaises. Les bilingues représentent 4 paires de langues: Penjabi-anglais, sujets d'origine indienne; Mirpuri-anglais, sujets d'origine pakistanaise; Italian-anglais; et Patois-anglais, sujets d'origine caribéenne. Des tests de raisonnement déductif et de résolution de problèmes en arithmétique représentent les principales variables dépendantes. Explorer l'influence de la compétence d'un enfant dans sa langue maternelle sur sa capacité à raisonner mathématiquement dans une deuxième langue (ici l'anglais) est un des buts principal de l'étude. On cherche d'autre part à établir quelles sont les variables linguistiques spécifiques-telles que la compréhension écrite ou la capacité à utiliser les mots d'articulation logique-qui peuvent rendre compte d'une grande proportion de la variance des résultats aux tests de raisonnement et de résolution de problèmes. En testant l'hypothèse de l'interdépendance linguistique proposée par Cummins (1979) sur plusieurs couples de langues, en espère éclairer d'un jour nouveau la question de l'éducation des enfants qui apprennent les mathématiques dans une deuxième langue.

In different countries throughout the world many people, both adults and children, are called upon to work or study in a language other than the one they would normally speak at home. Such a situation is well known in Europe and perhaps more recently has developed in North America. For example, many Canadians are finding it necessary or desirable to study through the medium of a second language. In Canadian schools many English children are taught through the principal medium of French - the so called "immersion" programs - while French speaking university students in various disciplines are required to work from textbooks written in English. In other industrial nations it is common to find well established minority languages, for example Welsh in Great Britain, Spanish in the USA and where immigrant communities have been established additional languages to that of the host country have increasing cultural and educational significance as the community grows and their children attend school. Again, in developing countries, formal education is often given in a European or a national language which is not that of the learner. Nowhere is this more evident than in Africa where both French and English still assume much educational significance. We see then that bilingualism has international importance when we begin to examine the problems surrounding the education of children who are required to learn in a second language. And, in particular, it is of great interest to us as mathematics educators to understand how this specifically relates to the teaching and learning of mathematics.

In Britain today there are large numbers of children learning mathematics in a second language. Many of these are children of immigrant families from India, Pakistan, Africa and the West Indies. The current debate about the provision of mother tongue teaching for such children has focused on the type of provision, how it could be introduced and more fundamentally why it should be supported. In seeking to answer such questions educators have turned to research for evidence on which to base decisions. Unfortunately it is precisely here that confusion has been increased by contradictory research findings. With respect to mathematics a recent extensive survey of the literature carried out for my own research has revealed little significant mathematical work to build on.

Up until the late 1960's most studies which either directly or indirectly dealt with mathematics, concentrated on school achievement as the dependent variable. In the main, they pointed towards a handicap for bilinguals in problem solving, but not arithmetic skills, when compared with monolinguals (Macnamara, 1966). The contrast between the performances in the two types of arithmetic is almost certainly a reflection of the difference in their dependence on language. As such, the results found are not surprising. However other findings (for a review see Austin and Howson, 1979) suggest that the problem hinges on partial linguistic mastery and that the apparent retardation is not absolute. Rather the results can be retrieved by those students who eventually become competent in the deeper structures of a second language and find it as easy to 'think in' as their first language - a phenomenon Macnamara (1970) calls 'grasp of language'. It seems however that relatively few bilingual children reach this stage and even among very competent adult bilinguals D'Anglejan et al (1979) found weaknesses in the more demanding cognitive activities involved in the retrieval and manipulation of stored information in the second language. Morris (1978) concludes that it is very likely that those who must learn mathematics in a second language are handicapped, and further, that this would carry over into scientific and technological subjects which use mathematics.

During the 1970's the quality of research in the area improved markedly. Studies were better controlled and earlier methodological shortcomings taken into account. Few were specifically concerned with mathematics but the general picture which emerged tended to reverse the trend of earlier investigations which pointed to poor academic achievement of bilingual children. Thus in certain circumstances bilingual children have been found to have superior cognitive flexibility, a more diversified set of mental abilities, and to show advantages on measures of creativity, divergent thinking and problem solving in science (for reviews see Cummins, 1976; Kessler and Quinn, 1980). Of particular mathematical interest was a study of Finnish migrant worker's children in Swedish schools (Skutnab-Kangas and Toukomaa, 1976). These researchers found that mother tongue development was a key factor for success in mathematics at school despite the fact that it was taught in Swedish. Thus a new theoretical line of thought began to emerge which tried to explain differential academic achievement of bilingual children in terms of the strength of their two languages, rather than blaming either bilingualism itself or linguistic mismatch per se for the observed academic retardation among many language minority children. A recent theory of linguistic interdependence in learning has been put forward by Cummins (1979). His work has emerged from the well documented success of 'immersion' programs in Canadian schools for English speaking children during the last decade, findings which cut across earlier research and which were clearly inconsistent with earlier theories.

My own research in Cambridge has been concerned with the effect of language variables on reasoning in mathematics and mathematical achievement at school. In particular, when the children are bilingual and at a stage of cognitive development where language is becoming increasingly important as a vehicle for classroom learning and communication of thought. The sample consists of 203 bilingual and 167 monolingual children in the late primary and early secondary years of English schools aged 11-14 years. Five language groups are represented: Punjabi - English speakers of Indian origin, Mirpuri - English speakers of Pakistani origin, Italian - English speakers, Creole - English speakers of Caribbean origin and monolingual English children. Some key questions to be answered are:

1. Does the child's first language competence have an important functional influence on his/her ability to reason in mathematics in a second language?
2. Do particular language pairs exert different effects?
3. Do 'additive' bilinguals (Lambert, 1977) outperform English monolinguals and other bilinguals on tests of reasoning and problem solving in mathematics?
4. What specific variables account for a major proportion of the variance in mathematics reasoning scores?
5. What light does the study throw on the current debate of 'cognitive advantage vs cognitive deficit' from a mathematical standpoint?

At this stage the data has been punched preparatory to analysis and I hope to be able to present some initial findings at the Conference. To help clarify this the major variables are given below. The test instruments were constructed by the researcher in English and validated in a pilot study.

A. Dependent Variables
   1. A test of deductive reasoning through linear syllogisms.
   2. A test of word problems in arithmetic.

B. Independent Variables
   1. Reading Comprehension in English (using a Cloze procedure to probe deeper language structures).
   2. A test of logical connectives in English (see Strevens, 1971) using a Cloze technique.
   3. A test of listening comprehension in the child's mother tongue.
   4. A non-verbal test of mathematical development related to Piagetian cognitive stage (Cornish and Wines, 1977).

These six tests were given to all children in the sample together with a questionnaire which enables one to either control for, or include as independent variables, such variables as age, sex, school, school achievement, home background variables, language usage and so on. Children were eligible for selection provided their teachers considered them to be literate in English and spoke the language fluently enough to be well understood in ordinary conversation. The mother tongue testing was carried out by native speakers of the languages concerned who were themselves teachers attached to the different schools.

Apart from the initial results, it is intended to review a number of problems encountered during the research and decisions taken concerning their solution. Of particular importance in this regard is a definition of bilingualism and how it can be measured.

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ASSIMILATION OF MODELS OF UNDERSTANDING BY ELEMENTARY SCHOOL TEACHERS

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Dans la première année d'un projet de quatre ans, nous avons cherché un moyen d'enseigner des modèles de compréhension pour les rendre accessibles aux maîtres du primaire. Après une étude préalable de ces modèles avec un groupe de 28 enseignants, il s'agissait, par un travail en petites équipes, de classer suivant quatre modes de compréhension (intuitif, instrumental, relationnel, formel) les interprétations recueillies au cours d'une discussion plénière préalable portant sur des notions telles que le nombre, l'addition, la soustraction, etc.

Les résultats obtenus par le groupe expérimental, comparés à ceux d'un groupe contrôle, montrent que la formation aux modèles de compréhension permet aux enseignants de découvrir un nombre bien supérieur de façons de comprendre une notion donnée. De plus, la réussite à deux tests de transfert chez le groupe expérimental indique que les enseignants peuvent appliquer ces modèles pour analyser des notions voisines. La réussite au test passé en équipes s'est avérée supérieure à celle du test passé individuellement.

D'autres effets psycho-pédagogiques importants, qui font l'objet d'une autre communication, ont aussi pu être observés (Herscovics, Bergeron, Nantais-Martin, 1981).

(Version française disponible auprès des auteurs)

1. INTRODUCTION

At last year's meeting in Berkeley, we introduced our four-year research project on the pre-service and in-service training of teachers. In this communication the results and problems encountered in our first year of experimentation are reported. In order to put it in context, we recall the major outline of this project.

Eventually, we wish to train teachers in the use of clinical methods such as the diagnostic interview and the teaching experiment. Such a training would allow them to go beyond the written work of the students and reach into their thinking and learning processes (Herscovics & Bergeron, 1980). However, some exploratory work has shown that such a training must be done within a frame of reference which will allow the teachers to analyse their observations as well as their pedagogical interventions (Bergeron & Herscovics, 1980). This justifies a prior

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training in the use of models of understanding and of learning models.

For this first year's experiment we set ourselves three main objectives: 1) to develop an appropriate teaching method for the models of understanding; 2) to apply these models to the analysis of mathematical concepts and algorithms used in the primary school program; 3) to elaborate methods of evaluating our teachers' progresses.

2. SELECTION OF SUBJECTS

In this first experimentation we have selected as our subjects a group of practicing teachers. This enabled us to benefit from their teaching experience and, their daily contact with pupils made it possible for them to verify immediately in their classroom the different modes of understanding under study. Our research was carried out within the framework of a regular in-service course offered within a program for the improvement of practicing elementary school teachers (Certificate in Mathematics and Science from the Université de Montréal). Twenty-eight teachers from the six grades of the primary school participated in this first mathematics course during a fifteen-week period and meeting once a week for three hours.

About half of our subjects were working at the first three grades and the other half at grades 4, 5 and 6. They averaged 14 years in scolarity and had taken courses in traditional algebra and Euclidean geometry. That these courses had been taken a long time ago is evidenced by the fact that they averaged 13 years of teaching experience.

3. TEACHING METHOD AND COURSE ORGANIZATION

The first part of the course was devoted to the theory, the second part to the application of this theory and, the last month was kept for the teachers' projects.
The first three weeks were assigned to the study of the Bruner (1960), the Skemp (1976), and the Byers and Herscovics (1977) models of understanding, as well as to some notions of Piaget's theory of development (stages, conservation, reversibility). Following this introduction, teachers had to analyze with the help of these models the concepts of number, numeral, addition, subtraction, zero, multiplication, division, positional decimal notation, and the algorithms for addition and subtraction.

Considering the difficulties we ourselves had experienced with such analyses and the time it took us (several days of reflection were sometimes needed for a simple concept), we could not expect teachers to succeed within a three-hour period. Moreover, in keeping with a constructivist philosophy of learning, we could not simply transmit our conclusions. Instead, we had to develop a teaching method which allowed the teachers to get involved.

The method we have conceived is characterized by the fact that it mobilizes the analytical resources of the whole group. At first, a general discussion of the question "What does it mean to understand this concept?" provided a great variety of answers. Following this, teachers were divided into small teams of 4 or 5 whose task was to classify these various interpretations according to the four modes of understanding: intuitive, instrumental, relational, and formal. Finally, the entire group was convened and each team reported on its classification.

4. EVALUATION OF RESULTS

Different means have been used to evaluate the extent to which teachers had assimilated the models of understanding.

4.1 PERCEPTION OF UNDERSTANDING BY UNTRAINED TEACHERS

In order to judge the effect of the teachers' training in the use of models of understanding, we have used a control group as a basis for comparison. The selected group consisted of 19 teachers working at the same elementary grades and with comparable teaching experience. Moreover, they were enrolled in the same program and were taking their second mathematics course. Two questions were asked:

1) "Are there several ways of understanding the addition of two natural numbers whose sum does not exceed 9?"
2) "Are there several ways of understanding the number zero?"
"If yes, give an example for each way".
"If no, explain what it means to understand this notion."

The evaluation of their answers was based on criteria discussed in the section on transfer tests. For addition, most of these teachers have interpreted "understanding" in terms of "modes of representation" which they had studied in their course. This explains why they focussed on different representations rather than on different ways of adding. The following table shows the number of teachers who have identified 1, 2, 3,or 4 modes of understanding.

no.of modes	0	1	2	3	4	total/76	%
question on +	0	6	8	5	0	37	49
question on 0	3	8	6	2	0	26	34

The percentages are based on the total number of modes that can be identified (76), namely 4 modes for each of the 19 teachers. These results cannot be considered as the outcome of individual reflections since the questionnaire was administered to a class of teachers who discussed it in small groups of three. Nevertheless, we can note that the teachers' perception of understanding is not uniform but varies according to the concept in question (49% for addition, 34% for the number zero).

4.2 PERCEPTION OF UNDERSTANDING BY TEACHERS ACQUAINTED WITH MODELS

We report here the results of the experimental group's analyses performed in teams and also those obtained from two transfer tests.

TEAM REPORTS

As described under "teaching method", following a general discussion of a given concept small teams of teachers had to report the various modes of understanding they had identified. For five of these concepts we have recorded each team's report. The following table shows the number of teams who have identified 0, 1, 2, 3 or 4 different modes.

no.of modes	0	1	2	3	4	total/24	%
number	1	1	0	2	2	15	62
addition	0	0	1	2	3	20	83
zero	0	0	0	3	3	21	87
numeration	0	0	0	1	5	23	95
+ algorithm	0	0	1	2	3	20	83

Percentages are based on the total number of modes which can be identified (24), namely 4 modes for each of 6 teams. As is shown for addition in the following section, more than one criterion can be used to characterize a given mode of understanding. As with the control group, we have considered an identification as acceptable if at least one of the criteria was mentioned. This can be justified by the fact that we are more interested in the multiplicity of the modes identified by the teacher (the number of different ways of understanding a concept) than in the number of criteria used to describe a single mode.

A comparison of the results between the experimental group and the control group (83% vs 49% for addition, 87% vs 34% for zero) shows remarquable differences. However, these cannot be attributed solely to the assimilation of the models of understanding. Indeed, two other variables must be taken into account. On one hand, the experimental group had the advantage of a general discussion whereas the control group was limited to discussions in groups of three. On the other hand, the general discussion lasted one hour for each concept while the small-group discussions took about 15 minutes. The time spent by the control group was restricted due to the fact that only a half-hour could be aloted within the other course.

TRANSFER TESTS

"Transfer" is defined as "a phenomenon by which progress achieved in the learning of a given activity brings about an improvement in a different but more or less related activity" (Piéron, 1979). We accept this definition but wish to extend it in the sense that transfer can also mean "that progress achieved in a given activity brings about the ability to use this activity in a different but more or less related activity.

Two tests verifying a possible transfer in the ability to use the models were administered. The first one, from addition to subtraction, dealt with individual transfer and the second test, from multiplication to division, dealt with transfer at the team level. The closeness of the related operations was warranted by the difficulties we had ourselves experienced in the analysis of the concepts.

The following tables show the results of the first transfer test and the criteria used to describe various modes of understanding. These criteria take into account various investigations of early arithmetic (Carpenter & Moser, 1979; Fuson, 1979; Nescher, 1979).

no. of modes	0	1	2	3	4	total/112	%
Addition	1	4	6	9	8	75	67
Subtraction	1	2	6	17	2	73	65

	Intuitive*	Instrumental**	Relational	Formal
ADDITION	-to add to -to bring together	-to count all -to memorize addition facts	-to count on -to relate to subtraction (5 ?= 9)	to be able to interchange symbolic with iconic and enactive representations
SUBTRACTION	-to remove	-to count remainder -to memorize subtraction facts	-to count back -to relate to addition (9 - ? = 5)	-same as above

* in the sense of unquantified action; ** in the sense of construction

We can consider here a transfer since only in the case of addition was there any teaching and not in the case of subtraction. We now explain how we arrived at an index of transfer. The 75 modes identified for addition, out of a possible 112 (4 modes for each of the 28 teachers), give us 67%. It should be noted that comparing this with the 65% achieved in subtraction (73/112) would give us an index of 97% (73/75). However, this would be misleading since the modes identified for addition were not always the same as for subtraction. By using for our ratio only those cases where they were the same (66) and the number of correct identifications for addition (75) we obtain an index of 88% (66/75). This index is comparable with the one obtained in the second transfer test.

no. of modes	0	1	2	3	4	total/108*	%
Multiplication	0	0	1	11	15	95	88
Division	0	0	1	7	19	99	92

* one subject withdrew leaving us with 27 x 4.

Since the number of modes identified in division which correspond with the ones in multiplication is 88, the ratio 88/95 gives us a transfer index of 92%. Two reasons can explain the better results in this second test (88%, 92%) as compared with the first one (67%, 65%). First we must recall that the multiplication/division test was administered to teams of teachers whereas the addition/subtraction test involved individual work. The other reason is related to experience. The concept of addition was only the third one analysed while multiplication was the sixth.

CONCLUSIONS

In the first experiment we wanted to find out if the models of understanding were "teachable" in the sense of being assimilable by our subjects. Our results show that the answer is affirmative. In fact, the experimental group was quite successful in identifying various modes of understanding for each concept.

The transfer tests show that teachers can apply tehir newly acquired analytic tools to closely related concepts. Their results indicate quite a difference between individual transfer and team transfer. This confirms our belief that a training in the application of the models ought to pool the group's resources and that our method of instruction was appropriate.

We think that this experience has benefited our teachers. In comparing their work with the control group, it seems clear that their perception of understanding has evolved. Both the team reports and the transfer tests show that a far greater multiplicity in the ways of understanding a concept were identified by the experimental group as compared to the untrained teachers. Other effects of a psycho-pedagogical nature have been observed and are reported in another paper by Herscovics, Bergeron, Nantais-Martin (Grenoble, 1981).

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SOME PSYCHO-PEDAGOGICAL EFFECTS ASSOCIATED WITH THE STUDY OF MODELS OF UNDERSTANDING BY PRIMARY SCHOOL TEACHERS

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Dans une première communication, Bergeron, Herscovics et Dionne (Grenoble, 1981) ont décrit leur méthode d'enseignement des modèles de la compréhension ainsi que les moyens employés pour déterminer le degré d'assimilation de ces modèles par des enseignants du primaire. Cette formation à l'analyse conceptuelle a eu des conséquences psycho-pédagogiques intéressantes.
Une approche didactique intégrant la mathématique, la psychologie, la pédagogie et l'épistémologie a provoqué certaines formes d'anxiété. Celles-ci n'ont été que passagères et plusieurs évidences indiquent que les enseignants ont changé leur perception de la mathématique, de leur propre compétence mathématique, ainsi que des processus d'apprentissage.
Comme en témoigne la tendance qu'ils ont acquise à se décentrer de la réponse écrite pour s'attacher davantage aux processus de pensée, les enseignants ont développé une perception constructiviste de l'apprentissage.

(Version française disponible auprès des auteurs)

In a first communication, Bergeron, Herscovics and Dionne (Grenoble, 1981) have described how they taught models of understanding to primary school teachers and the means of evaluation used to assess the degree of assimilation they achieved within a regular 45-hour certificate course. Such a training in conceptual analysis has had a number of interesting psycho-pedagogical effects which are the topic of this second communication.

MATHEMATICAL COMPETENCE OF OUR TEACHERS

A first effect relates to changes observed in the experimental group at the mathematical level. Judging by the number of courses taken up to their initial training, their content, and the time elapsed since the last course, the mathematical competence of our teachers was rather limited. This led us to discard the standard "pretest-test" format in order to avoid the anxiety and the negative attitude generated by failure at such tests. However, several signs point to a weakness in mathematics. For instance, we note that in their analyses the teachers were often confusing notions such as area and surface, number and

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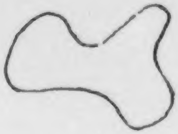
numeral, the operation of addition with its algorithm.

The confusion about area is particularly revealing. Right at the beginning of the course, we had passed around a questionnaire on this topic. The questionnaire was not meant to be used as a pretest. It was merely to allow the teachers to follow the changes in their own thinking while learning about the models of understanding which were illustrated by being applied to the notion of area. The following question was asked:
"How would you go about estimating the area of this figure?"



Of the 27 teachers who answered, 14 indicated that they would cover the figure with square units whose number would give them an approximation; two teachers suggested enclosing it inside a rectangle and then estimating visually the proportion taken up by the figure; six others would have measured the contour of the figure with the help of a string which they would use to construct a rectangle with the same perimeter and then apply the formula for the area of a rectangle; finally, five teachers indicated that they simply had no idea. Remarkably, 8 of the last 13 teachers mentioned were working at the upper primary grades (4,5,6) in which the concept of area is taught. The discussions on area within the context of modes of understanding led all our subjects to master this notion. Of course this cannot be attributed solely to the models since a purely mathematical discussion would probably have yielded the same mastery. However, such a discussion would not have brought about a reflection on the cognitive aspect.

By focusing the teacher on the cognitive aspect of learning, we think that we have changed his perception of his own mathematical competence. Indeed, in an anonymous questionnaire handed out at the end of the course, 25 out of 28 participants thought they had achieved a better understanding of mathematics.
This is somewhat surprising for we did not expect people who had taught "elementary" notions such as number, addition, subtraction, etc. for so many years, would feel, after a few weeks of analyses, that they themselves had attained a better grasp. We suggest as a possible explanation that the analysis of understanding, not only changes their perception of their own competence, but also their perception of mathematics. In fact, the models assign to mathematics cognitive dimensions which go beyond the current instrumental and formal interpretations. Through their analyses, teachers have to reconstruct mathematics in a psycho-genetic context which valorizes the intuitive and relational aspects.

CAUSES OF ANXIETY

Despite our caution, not all forms of anxiety could be avoided. Indeed, during the first five weeks, reports from school consultants and private comments by teachers revealed that some of them were ill at ease with the required task of analyzing concepts, and that such demands could shake their initial self-assurance. This can be attributed to several factors.

First of all, the teachers were expecting a regular course in mathematics which would not over-emphasize the psycho-pedagogical aspects. Then, there is the fact that a long established tendency to evaluate only the written answers hampered the acceptance of a new form of evaluation based on the processes used. This is clearly shown by the oft-repeated question: "How can one be sure that this understanding is not instrumental?". It took five weeks for the teachers to realize that the criteria used to determine these processes are generally mere indications and not proofs. Indeed, nothing can guarantee that the procedures used are not the result of instrumental learning. Only through an appropriate questioning can it be determined.

Another reason for anxiety was sometimes due to differences between the treatment of a concept in the textbooks used by teachers in their own classroom and the treatment suggested by the conclusions they had reached during discussions in the course. For instance, their textbook treated addition in terms of states (the states preceding and following the union of two sets) while ignoring the operator interpretation (the natural act of adding to).

The in-depth analysis of a mathematical concept brings out its great complexity which while putting it under a new light ,also can become a very justifiable cause of uneasiness. For example, the concept of number perceived as a measure of a quantity of discrete objects (Vergnaud,1979) raises questions about units of measure (Steffe et al.,1981), about various ways of counting (Comiti et al.,1980), about conservation of number (Piaget,1965), as well as problems associated with its symbolic representation (Ginsburg,1977).

A last source of anxiety that we have been able to identify is related to the difficulty of interpreting the modes of understanding. We had chosen several criteria to describe a given mode and many teachers gathered them under other classifications. For instance, in the transfer tests, 13 out of 28 subjects confused some of the modes, the greatest confusion appearing between the intuitive and instrumental modes,and between the relational and formal modes. The following table shows the nature and the distribution of the mistakes:

	ADDITION				SUBTRACTION			
Mode	Int.	Instr.	Relat.	Formal	Int.	Instr.	Relat.	Formal
Int.		5	3	0		4	3	1
Instr.			1	3			1	2
Relat.				5				9

Note: The sum of these numbers exceeds 13 since the same teacher can be mistaken several times.

We ourselves probably contributed to this confusion. As a matter of fact, we used unquantified action as a criterion for intuitive understanding while extending Skemp's definition of instrumental understanding ("rules without reason") to include a first quantification of the previous action, without communicating this extension explicitly enough. As regards the confusion between relational and formal, it was due to the fact that several teachers had interpreted "relational" to mean the "relation" between the symbolic expression and its enactive and iconic representations.

In retrospect, we find that we have tried to adapt the tetrahedral model (Byers & Herscovics, 1977) to the construction of concepts. As a matter of fact, as mentioned earlier, we had included a "first construction" in the instrumental mode, had associated reflective abstraction with the relational mode, and ended up identifying the formal mode as a "formalization" of relational understanding. Following discussions with Skemp (1981), we have realized that the model we are developing should allow us to differentiate between understanding considered as a state and understanding considered as a path on which one treads slowly.

DE-EMPHASIZING THE ANSWER

In our first experiment, our aim was to instruct teachers in the analysis of understanding in order to provide a framework for their eventual training in clinical methods. These methods would then allow them to de-emphasize the so-called "products" of learning (the written answers) and instead, focus on their students' thinking processes. However, we have some evidence showing that such a de-emphasis can already result from the use of models of understanding in the analysis of concepts.

A first indication comes from the analyses done in teams. Table 4.2 of the previous communication by Bergeron et al.(1981) shows that many teams were able to identify four modes for several of the concepts dealt with. This means that they were able to discriminate between the instrumental and relational modes which often differed in terms of procedures (cf. addition: to count all or to count on). Thus we note that, due to the models, teachers begin to take into account the processes used.

As a second sign we have the projects which were carried out by the teachers within the course. Small teams varying from three to five teachers followed the evolution of a given concept or algorithm throughout the different primary grades by interviewing from one to three children in each grade. And, it is on the basis of the children's thinking that all the teams evaluated their understanding.

Finally, patent indications were provided by the answers to two questionnaires handed out in the last weeks of the course. To the question "Do you think that the use of models of understanding in your teaching is desirable?" all the answers (18 replies) were affirmative and the following justifications were given: "allows us to follow the child in his reasoning"(5); "allows us to think about the way we teach, to evaluate and help the children"(5); "yes, for we were asking only for one mode of understanding (instrumental)"(3); unjustified "yes"(5).

These answers confirmed those obtained two weeks before to the question: "Has this course led you to improve your teaching of some mathematical concepts? -If yes, can you describe the changes in one of your lessons (for example, in the preparation, in questioning of the students, in the interpretation of the feedback, in the evaluation of the written work, etc.)?". Out of 25 teachers 24 reported an improvement in their teaching relating it to the lesson (13), to the questioning (17), to evaluation (13), or to remediation (10). The following comments provide some of the flavor:

- "When I now prepare a lesson I try to reach the four types of understanding with the child."
- "Often when correcting the exercise book I even ask the ones who have everything good."
- "For subtraction I had 5 out of 25 pupils who knew the why for the 1 we borrow. I couldn't get over it."
- "I try as often as possible to have the child explain his procedure and I don't take his mistakes as a sign of a total lack of understanding."

CONCLUSIONS

By raising the question "What does it mean to understand such a concept?" we have encouraged the teachers to consider the psycho-pedagogical aspect of the teaching of mathematics. Even if this has made them aware of the complexity of the notions they taught, they did not feel that their own mathematical knowledge was being examined. In light of their weakness in this subject, a direct examination might have provoked mathematical anxiety. Although other forms of anxiety were induced, these were of a temporary nature and we have several indications showing that the course has brought about changes in their perception of mathematics and of their own competence in this discipline. These changes, resulting from an approach which integrates mathematics, psychology, pedagogy and epistemology, are particularly important since they provide the teacher with a constructivist viewpoint of the learning process.

That the teachers develop such a constructivist perception of the learning process has been shown by their tendency to de-emphasize the written answer. This is evidenced by the increased importance they attach to the questioning of the pupils as indicated by the comment "Often when correcting the exercise book I even ask the ones who have everything good." The idea expressed by this teacher is most interesting: not only is she questioning the students who make mistakes, but she also questions those "who have everything good". This remark underlines the complementary roles played by the written test and by the questioning. If the former allows for the evaluation of skills, only by an appropriate questioning can one evaluate the process used.

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The Nature of Geometrical Objects as Conceived by Teachers and Prospective Teachers

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Dans les classes de géométrie de l'enseignement secondaire, on insiste souvent sur 1. la nature déductive de la géométrie
2. la nature des objets géométriques.
La question de la nature des objets géométriques peut être traitée de bien des façons et elle a soulevé de nombreuses controverses chez les mathématiciens. Dans la perspective de ces controverses, il est intéressant d'étudier comment les enseignants et les futurs enseignants de mathématiques conçoivent cette question. Bien qu'ils soient le plus souvent ignorants des grands débats philosophiques, on retrouve chez eux des philosophies personnelles, implicites, à la fois naives et complexes. Ce sont ces "philosophies spontanées" que nous avons tenté de mettre au jour.

Des questionnaires ont été distribués aux enseignants et aux futurs enseignants. L'analyse des réponses a permis de dégager trois conceptions principales:

1. *Les objets géométriques sont des objets réels; ces objets font partie de notre espace. Certains sont visibles, d'autres pas.*
2. *La géométrie s'occupe de déterminer quels sont les énoncés qui peuvent être déduits d'un ensemble de postulats. Il n'y a donc pas lieu de se préoccuper de la nature des objets géométriques; ils peuvent être conçus comme des objets abtraits. Toutfois, de tels objets n'existent pas.*
3. *On accorde aux objets géométriques différents status existentiels. Ils existent en théorie, dans l'univers des abstractions ou dans l'esprit du sujet.*

§1 The Problem

The question of the nature of the mathematical objects is mainly a philosophical one. Nevertheless, it finds its way to the mathematical curriculum and to Math textbooks. Consider, for instance, the question what numbers are. One of the answers to it is that numbers are abstract objects that have many different representations. Hence, there should be a distinction between numbers and numerals (number names). It is taught very often at the elementary level to second or third graders. It is also one of the reasons (although not necessarily the only one) for including enumeration systems in the junior high curriculum.

The question of the nature of geometrical objects has a slightly different character (note that we are discussing geometrical objects as mathematical objects and not as physical objects). At the elementary level, where only some geometrical figures are introduced to the students, it does not rise because the geometrical object is identical with the geometrical figure drawn at the book or elsewhere.

On the other hand, at the junior high level or at the secondary level, when Euclidean geometry is introduced (if at all) as a deductive theory, it is raised, in one way or another, by textbooks or teachers. The student is told that the geometrical points and lines are different from those which are drawn in the books or on the blackboard. The straight line is infinite and between any two of its points there is another point. The point has no dimensions and so on. Later on, in most cases, this question is ignored or even forgotten and it is not clear whether it has any importance to the student's geometrical development. On the other hand also the contrary is not clear. Our starting point is that whenever cognition faces new objects it seeks an answer to the question what these objects are. Part of this answer classifies the reality to which these objects belong and another part determines the relations between the objects and their reality or between the objects and themselves. (Those who watch children can tell how they know to distinguish between physical reality, fairy tale reality, T.V. reality and so on. Also how they form realities for microbes, molecules, numbers and so on.) This cognitive process is spontaneous and in many cases not verbal. The person concerned is very often verbally unaware of it. We are interested in the question what types of reality people associate with geometrical objects. As we just said this might be implicit and non-verbal and if this is so how can we find it out? There are two answers to that. First, we observe behavior and raise some hypotheses about the (implicit) concepts that might form such a behavior. Second, when being questioned, the subject, sometimes, can make the unconscious-cousclous the non-verbal-verbal and the implicit-explicit. The chance for this to happen is greater if the subject is at the threshold of verbality.

§2 The Sample and the Questionnaires

Our aim was to inquire naive approaches to geometrical objects. Therefore we chose subjects who (in most cases) did not have formal training in the philosophy of Mathematics. On the other hand, they are very likely to develop (implicitly or explicitly) personal, "homemade", philosophies about the nature of mathematical objects in general, and particularly about the geometrical objects. The sample consisted of 2 groups. The first one included 18 prospective teachers (who were students in their second or third year of studies toward a B.Sc. degree in Math or toward a teaching certificate). The second group included 29 Math teachers at the junior high the senior high levels. Two questionnaires, based on informal discussions with students, were compiled. They were administered to the above sample in 1978, 1979. The questionnaires were:

QUESTIONNAIRE I: 1. In the introduction to a Geometry textbook was written: "The points we draw have a notable size so that we can see them. But real points are so tiny that they cannot be drawn by any pencil even if we could make it as sharp as we wish". In addition to this also the following was written: "We all know what straight line is and how it looks like. This is a drawing of a straight line: ———————— Of course, it is not a real one because a real line has no width. A real line cannot be drawn". In a homework assignment, given after this had been taught, one student wrote: "It follows from the text that points and lines cannot be seen or sensed. From this point of view they are like molecules. Also molecules cannot be seen or sensed. However, they exist in the world and scientists inquire their properties. In a similar way, mathematicians inquire the properties of points, lines and other geometrical shapes".

As a prospective teacher (or teacher) what will you write in the student's notebook: A. Correct B. Incorrect C. Something else (please specify). Add a comment to explain your view.

2. Another student wrote: "Everything in the world has width, even small particles as molecules and atoms. If points and lines do not have width it follows that they do not exist. Therefore, Geometry inquires things that do not exist".

What will you write in the student's notebook: A. Correct B. Incorrect C. Something else (please specify). Add a comment to explain your view.

QUESTIONNAIRE II: 1. In your opinion, is there any difference between an abstract object and an imaginary object? A. Yes B. No. If yes please specify.

2. Is every abstract object is also an imaginary object? A. Yes B. No.

3. Are there abstract objects which are also imaginary objects? A. Yes B. No. If yes please note one or two.

4. The geometrical figures are: A. real objects in the space, although it is impossible to see them. B. abstract objects. C. imaginary objects. D. something else (please specify)

The second questionnaire is meant to clarify the status of the geometrical objects in a more direct way. It also expresses two different quite common approaches to mathematical objects. The first one assign to geometrical objects an existence which is somehow independent of our mind. The second one considers them as something formed by our mind and in this sense they are a sort of imaginary objects. Abstract objects, according to that, will be a special case of imaginary objects, whereas according to the first approach they are not.

§ 3 The Results

The answers to the questionnaire were analysed and classified. The considerations leading to our classification cannot be given here, however, the interested reader will (hopefully) be able to reconstruct them in the specific cases that will follow. Both the answer profile and the written comments were used to determine the category of the subject. Sometimes, the differences between the categories are very small and sometimes the subjects are inconsistent, having elements from conflicting catgories (something which we consider quite natural in suc a complicated situation). In case of inconsistencies the category was determined according to the dominant elements. The answer profile will be denoted for instance by *I1A2BII4B* (which means that the answer to question 1 in questionnaire *I* was A, the answer to question 2 was B and answer to question 4 in questionnaire II was B). The words "prospective teacher" will be denoted by PT and the word "teacher" wil be denoted by T.

Category I: The Geometrical objects are real. They are part of our space.

I1A2BII4A(PT): It is possible to inquire also things which are not concrete but nevertheless they exist in space like points and lines

I1A2BII4A(PT): These things exist but they cannot be drawn accurately on paper.

I1C2BII4A(PT): I would introduce the point as the intersection of straight lines since straight lines can be seen in space (for instance the sides of a cube).

I1C2BII4A(T): Partially correct because many lines together form a surface which is real.

I1A2BII4B(T): There is no need to waste time on concepts having no importance at all. One should explain that a point is somethin very thin and so is the line. The more we draw them accurately the closer we approach reality. .Abstract objects exist whereas imaginary objects do not exist.

Category II: Geometry only inquires what can be derived from certain hypotheses, geometrical objects are idealizations that do not exist, abstract objects do not exist. Geometry is only a tool by means of which we can inquire the real world.

I1B2AII4B(PT): Molecules exist in reality, they are part of nature whereas points and lines are abstractions of reality; they are idealizations. Lines and points as presented in the questionnaire do not exist at all. However, in a less "ideal" form there are lines in reality. The inquiry of the idealizations helps us to understand reality.

I1B2AII4B(T): Geometry does not deal with objects that exist in nature. As a matter of fact it does not inquire things. It is a means to solve problems (including problems in science).

I1A2AII4A(T): The student is very realistic. He lacks imagination but I don't blame him. He was not taught that we can work in the "as if" method ...Imaginary objects emerge from imagination. As to abstract objects-you start with something that exists and you disregard some of its properties until you reach the abstract object.

Category III: There are two worlds-the concrete and the abstract. The geometrical objects exist in the abstract world or in theory or in the human mind or imagination.

I1B2BII4B(T): It is true that points and lines do not have concrete existence. However, they exist in theory or as abstract objects. One has to emphasize the abstract existence. Not only things which you can see or touch exist.

I1B2BII4B(PT): There are several modes of existence. One can also inquire things which exist only in theory.

I1B2BII4B(PT): They exist in the human minds.

The statistical results are shown in table 1.

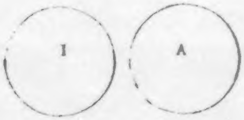
Table 1

	Category I	ategory II	Category III	Total
PTs	4(22%)	4(22%)	10(56%)	18
Ts	14(48%)	3(10%)	12(41%)	29

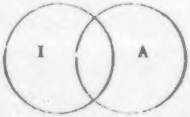
If categories II and III are considered together (as categories which are "professionally acceptable") and χ^2 is used then there is a significant difference ($p<0.07$) between PTs and Ts. One possible reason for that is that the more one teaches geometry the more he believes that geometrical objects are part of reality. However, there might be also alternative explanations.

As to the relations between abstract objects and imaginary objects there are 4 cognitive maps involved (I is for imaginary and A is for abstract).

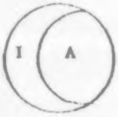
Map 1



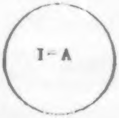
Map 2



Map 3



Map 4



(A typical example of an imaginary object which is not abstract was a witch. Something like that was mentioned by several students who claimed that imaginary objects and abstract objects were different.)

There was no significant difference between the two groups, hence the results are given in one table.

TABLE 2

	Map 1	Map 2	Maps 3 and 4
N = 49	35%	41%	24%

§4 An Educational Comment

The variety of approaches that exist in teachers (and also in professional mathematicians) suggest that there is no point to teach any of those at the junior high or at the senior high levels. According to the assumption in §1 the students will anyway develop a naive philosophy about the nature of geometrical objects. It is better not to interfer with this spontaneous and autonomous process. This principle of non-interference was advised also in Vinner, 1978 and Vinner and Tall, 1981. Of course, it is worthwhile to draw teachers' attention to this problem.

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PROVOCATIVE TEXTS AND SPONTANEOUS REACTIONS OF TEACHERS - A METHOD FOR RECOGNIZING TEACHING AND LEARNING OF MATHEMATICS

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Le sondage est de la plus haute importance dans le secteur des recherches empiriques de la formation mathématique. Les questions résultent des structures déjà existantes d'un sujet, p.ex. de la proportionalité. Mais est-ce qu'il y a un phénomène homologue à la proportionalité? Le meilleur contexte ou plus réel serait peut-être "fonction" ou "isomorphisme".
Pour reconnaître des domaines d'enseigner et d'apprendre des expériences on peut faire usage de textes provocants permettant des réactions spontanées et de grande portée des personnes interviewées.
Ces questions sont à pourvoir d'un titre provocant, d'un ensemble de théorèmes avec des mots-clé importants et d'une grande portée et d'un passage de texte s'adressant au professeur comme expert.
L'analyse de l'ensemble de ces réactions spontanées donne deux sortes des variables:

a) L'étendue des sujets existants en fait dans la procédure d'enseigner et d'apprendre.
b) Formes de sujet, p.ex. structures et caractéristiques d'enseigner et d'apprendre ces sujets, information de difficultés et avertissements concernant la procédure d'enseigner et d'apprendre.

Les résultats peuvent être utilisés de former les idées directrices pour la recherche ou pour servir de contrepartie en face des résultats déjà publiés dans un tel domaine.

AIMS OF QUESTIONING TEACHERS

Teachers influence pupils by teaching and see the everyday learning of pupils. Teachers are influenced by didacticians, textbooks, administrations, society.
Therefore questioning teachers can give information about the everyday teaching-learning process under realistic circumstances and restrictions. Information can be used for further exploration or for validating results of research.
Questioning must be aimed at important problems of everyday schooling, it must be voluntary and somewhat "provocative".

STRUCTURE OF TEACHER ORIENTED QUESTION TEXTS

Useful texts should be composed of a heading, some sentences with cues belonging to educational problems in the field aimed at, some sentences addressed to the teacher as an expert.

The heading should be an important and actual issue, not directed to the teacher himself but to his teaching problems.The heading can be a question or a statement; it must be most familiar to the teacher.

The sentences with didactical cues shall open a wide range of reactions. Therefore the cues must belong to the daily experiences of the teacher but they should not be too well sorted yet. It is useful to give chains of cues so that the teacher has a lot of cognitive and affective calls while reading the text. The cues should represent both the practical and the didactical aspects of teaching. Then the teacher will react as practitioner as well as curriculum expert.

The last set of sentences should be divided into four subsets. First the teacher should be invited to give spontaneous reactions - not to all cues and questions raised but to the "important" ones. The choice then made by the teacher is very important for analyzing the answers.

Second there should be sentences inviting the teacher to give hints for successful teaching and learning because he is an expert.

Third the teacher should be confirmed that he will be given the results of questioning.

And last not least it is very important to set the teacher free from essay styled answers. And do not forget to add an empty sheet of paper!

QUESTIONING

You can make the questioning by using a widespread teacher's bulletin or by a teachers' organization.

Questioning should not be made by using educational departments, governments etc. In this case teachers will react as conformists or non-conformists but not as teachers teaching pupils and not as experts for pupils' learning.

The questioning must be informal and voluntary.

The author's experiments show that about 5% of addressed teachers will react. Answers have a volume of 1/2-2 pages. About 90% of the answers are on a high level reflecting practice. They do not reproduce didactical theories.

ANALYSIS OF THE REACTIONS

Because the question texts are not quantitative qualitative variables must be searched for.
Above all there are two sorts of variables - dependent on one another - coming out of the answers:

(1) The RANGE VARIABLES

The answers show ranges of topics really existing in the teaching-learning process. You will see the range of topics, topics separated from one another and topics connected with one another.

(2) The SHAPE VARIABLES

The answers show for each topic
- structures and characteristics of teaching and learning in the topic
- difficulties of teaching and learning in the topic
- hints for teaching and learning.

Only in very few cases you will find hints for changing the ranges of topics or for restructuring the network of the topics. Teachers seem to be "topic experts".

WHAT CAN BE DONE WITH THE QUESTIONING RESULTS?

The results can be used as a first exploration of an unknown field of teaching and learning. Then headings and cues must be taken of this field. The range variables and shape variables found set up new foci for research.

In another way the results can be a counterpart for research already done in a field of teaching and learning. Then .you will choose the headings and cues
- for a special aspect of the field not yet very well known
- or for getting information about the validity of the shape of the field already known by research
- or for getting information about the network of topics in a widespread field etc.

The results can also be used for curriculum revision, in-service-training of teachers (Sturgess, 1980).

EXAMPLE: " FRACTIONS OR DECIMALS?"

The questioning was done in 1980. The text was published in a mathematics teachers journal. Out of 2 000 teachers 67 answered. The results were published in the same journal (Andelfinger, 1980).

The heading was: "Fractions or decimals?"

The most important cues were:

(a) pupils
  - reject computation with fractions
  - like decimals
  - are experts in decimal computation
  - are experts in decimal computation with pocket calculators.

(b) for daily life
  - fractions are unnecessary
  - decimals are good.

(c) mathematicians
  - need fractions
  - have many ways to deal with fractions.

(c) curriculum
  - brings fractions too early
  - brings fractions too late
  - brings decimals too late
  - brings decimals after fractions.

ANALYSIS OF THE REACTIONS

Some main results will be given here (for further results see: Andelfinger, 1980).

(a) Range variables

You can see that fractions and decimals are widely separated topics. They have nearly no common problems and difficulties. Proportional problems and problems of some other topics are noticed by the teachers but not as very important problems for fractions and decimals.

Obviously there are two ranges, fractions and decimals. They are only in loose connection with one another and with other topics.

(b) Shape Variables

- the image of fractions for teachers and pupils is negative; the image of decimals is positive.
- difficulties in teaching and learning of fractions are:
  - first fraction computing comes too early
  - kids cannot compute easily with natural numbers
  - kids do not have a feeling for the order of numbers
  - the building of formulas, definitions and algorithms for fraction computing is too big
  - didactical methods (e.g. operators) stop the idea of the number line; therefore pupils have no idea of fractions
  - kids have problems to see fractions as quotients
  - computing fractions in algebra is different from computing fractions in arithmetics
  - kids cannot see a pair of numbers as one number
  - "external" fractions (between measurements) and"internal" fractions are two different types of fractions for pupils.
- there are nearly no difficulties in teaching and learning of decimals excepted the problem how to put the decimal point.
- for hints see: Andelfinger, 1980.

USE OF THE RESULTS - TWO EXAMPLES

(a) An example for explorative use

It seems to be important how fractions and decimals are separated and connected in the mind of pupils.

Further research for these connections is needed.

(b) An example for use as a counterpart to research results already available

Teachers stated that pupils had had problems to see fractions as quotients.

Kieren shows that embedding fractions in a quotient field is very important for gaining the idea of rational numbers (Kieren,1975).

Suarez says that it is very difficult for most pupils to get the idea of rational numbers (Suarez, 1977).

When we compare our results to those of Kieren and Suarez we understand one difficulty for pupils in comprehending rational numbers. Obviously partitioning is not the same as making a quotient.

The results of questioning give a lot of hints for curriculum revision on secondary level, e.g.:

- natural numbers and decimals should be teached in connection with one another
- for pupils in the beginning of secondary level decimals are not fractions (except 0.5) and fractions are not proportions or quotients.

  Therefore the way to the concept of rational numbers cannot follow the "linear" mathematical way $N \longrightarrow Q^+ \longrightarrow Q$.
  Learning of rational numbers is a spiral way combining more and more the different aspects of rational numbers for many years.

REMARKS

Further questioning was done in 1980. The headings were:

- "Applied mathematical problem solving - math, folklore or what else?"
- "Rule of three - very easy". (see References)

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CONFERENCES PLEINIERES

(dans l'ordre du programme)

QUELQUES ORIENTATIONS THEORIQUES ET METHODOLOGIQUES DES RECHERCHES FRANCAISES EN DIDACTIQUE DES MATHEMATIQUES

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Research has developed in France along specific lines. This specificity is due to our philosophical and scientific traditions. For instance, the concept of "epistemological obstacle" (Bachelard) is hardly known outside France; the work of Piaget is not interpreted in the same way: although Piaget's work is our main reference, we are not satisfied with some of his views, for instance about stages, about mathematical knowledge, about logicism. New theoretical concepts are presented such as conceptual field, theorem in action, relational calculus, didactic variables and effects, system of signifiers, didactic contract. Also some methodological issues are analyzed, the stress is laid upon the necessity of using a variety of methods, and upon the importance of experimenting in the class-room. Class-room experiences require several conditions for them to be fruitful and repeatable. Illustrations of the theoretical conceptsand the methodological views are given and will be explained orally in greater detail.

Les recherches se sont développées en France, depuis une quinzaine d'années, sur des bases théoriques et méthodologiques sensiblement différentes des bases sur lesquelles elles se sont développées dans les autres pays. Cela tient principalement à des traditions philosophiques et scientifiques différentes, dont je donnerai deux exemples:

- des philosophes des sciences très influents en France comme Gaston Bachelard et Georges Canguilhem sont presque inconnus à l'étranger, en tout cas des professionnels de la recherche en éducation mathématique.
- les chercheurs français tirent de l'oeuvre de Jean Piaget des leçons différentes de celles qu'on en retire en général dans les pays anglo-saxons.

Pourtant nous partageons avec de nombreux chercheurs dans le monde les préoccupations suivantes:

- comment les connaissances mathématiques se développent-elles, chez l'enfant et l'adolescent en particulier?
- à travers quelles situations les concepts et les procédures mathématiques prennent-elles leur signification?
- quelles conditions didactiques et psychosociales faut-il rassembler pour assurer la transmission et l'appropriation du savoir, c'est-à-dire la reconstruction du savoir par celui qui apprend.

Certaines de ces questions ne sont pas nouvelles et l'on connaît par exemple la diversité des réponses apportées par les psychologues à la première question. Bien que, parmi ces réponses, l'approche de Jean Piaget nous apparaisse la meilleure et la plus féconde, nous n'en sommes pas satisfaits:

1. Jean Piaget s'est désintéressé de l'acquisition des connaissances scolaires. Il a plutôt cherché à caractériser le développement des instruments généraux de pensée, qui lui apparaissent relativement indépendants des connaissances scolaires.

2. Jean Piaget s'est intéressé davantage aux structures pouvant caractériser un stade donné de développement qu'à l'évolution adaptative des connaissances dans une situation ou un ensemble de situations où elles sont fonctionnelles.

3. Jean Piaget a séparé d'une manière exagérée la connaissance mathématique et la connaissance de la réalité physique. Je pense notamment à ce qu'il a écrit à plusieurs reprises sur l'abstraction simple et l'abstraction réfléchissante, la première portant selon lui sur les propriétés des objets et étant à ce titre constitutive de la physique, la seconde portant sur l'action du sujet sur les objets et étant spécifiquement mathématique.

4. Jean Piaget a privilégié les opérations et les structures logiques et contribué ainsi à minimiser les contenus de connaissance, que ces contenus relèvent de la physique ou des mathématiques.

Ces difficultés et l'insatisfaction générale dans laquelle nous laissent les recherches psychologiques sur l'apprentissage, nous ont conduit à définir de nouveaux cadres théoriques:

CHAMP CONCEPTUEL

La première grande question concerne les choix à faire pour découper à bon escient les contenus de connaissance mathématiques et en étudier de manière féconde la didactique et l'acquisition. Il n'est pas raisonnable d'étudier séparément l'acquisition de concepts (et de procédures) qui, dans les situations rencontrées et traitées par les élèves sont difficilement dissociables. Par exemple il serait aberrant de conduire des études séparées pour l'acquisition des concepts de multiplication et de division, de fraction, de rapport et de nombre rationnel, de fonction linéaire et n-linéaire, d'analyse dimensionnelle et d'espace vectoriel, puisque dès les premiers problèmes de type multiplicatif qu'il rencontre (proportions, surfaces, volumes), l'enfant est confronté à des relations qui relèvent de l'ensemble de ces concepts.

D'autre part, il est raisonnable, si l'on veut étudier la psychogenèse des contenus de connaissance, de découper la connaissance en domaines assez larges pour pouvoir en étudier l'évolution chez l'élève sur une assez longue période de temps et à travers un ensemble de situations diversifiées. La psychologie génétique nous apprend en effet que les connaissances se développent lentement; cela est vrai pour les contenus de connaissance comme pour les instruments logiques de la pensée.

Ce sont ces deux préoccupations (interconnexion des concepts et évolution psychogénétique) qui m'ont conduit à définir la notion de champ conceptuel: un champ conceptuel est un espace de problèmes ou de situations-problèmes dont le traitement implique des concepts et des procédures de plusieurs types en étroite connexion. Les deux champs conceptuels auxquels je me suis personnellement le plus intéressé sont les "structures additives" d'une part, les "structures multiplicatives" d'autre part. Evidemment ces deux champs conceptuels sont en relation l'un avec l'autre et en relation avec d'autres champs conceptuels comme l'espace, la dynamique, la logique des classes, etc. Mais ils m'apparaissent avoir une unité suffisante pour justifier des études distinctes. Ils me permettront d'illustrer brièvement la notion de champ conceptuel:

- exemple des structures additives: les premiers principes concernant l'addition et la soustraction, notamment l'augmentation et la diminution d'une collection et l'itération + 1 commencent à être appréhendés dès l'âge de 3 ou 4 ans, ainsi que l'a bien montré Rochel Gelman. Mais le développement des structures additives passe ensuite par le comptage, la conservation des quantités discrètes et continues, la numération, le traitement de relations différenciées (addition de mesures, transformation d'une grandeur, comparaison, composition de transformations, etc.) dont les plus complexes ne sont pas comprises par la majorité des élèves avant l'âge de 15 ans. On peut conduire, dans le champ conceptuel des structures additives, des analyses hiérarchisées sur la difficulté relative des différentes classes de problèmes, sur la complexité conceptuelle et la disponibilité des différentes procédures de traitement d'une même classe de problèmes, sur la signification et l'utilisation des différents systèmes de représentation symbolique. Il est maintenant démontré que l'acquisition des structures additives s'étend sur une période du développement de l'enfant et de l'adolescent supérieure à dix années, et passe par un réseau de chemins qui n'a que des rapports lâches avec la théorie des stades de pensée. On peut repérer dans cette acquisition, des sauts qualitatifs importants (comme celui de l'inversion d'une transformation) ainsi que des "obstacles épistémologiques" durables comme celui de la composition et de la décomposition de fonctions, ou comme celui de l'identification de l'ensemble des nombres naturels avec l'ensemble des nombres relatifs positifs.
- exemple des structures multiplicatives: Gerald Noelting a montré que la même situation de comparaison de rapports pouvait être utilisée entre l'âge de 2 ou 3 ans et l'âge de 17 ans et qu'entre ces deux âges se succédaient une dizaine d'étapes différentes. Le champ conceptuel des structures multi-

plicatives intègre le concept de rapport et le relie aux concepts de proportion et de fonction linéaire, ainsi qu'aux concepts de fraction et de nombre rationnel. On peut aisément repérer, dans ce champ conceptuel, des relations particulièrement difficiles, comme celles de produit et de quotient de dimensions que l'enfant rencontre dans des problèmes réputés élémentaires. On peut également identifier des procédures plus disponibles que d'autres, comme celles qui utilisent les propriétés de l'isomorphisme f(x)=ax, ainsi que des représentations symboliques plus naturelles que d'autres, comme les tableaux de proportionnalité simple ou double. Des "obstacles épistémologiques" particulièrement difficiles à surmonter existent, notamment dans la construction de l'ensemble des nombres rationnels, et dans l'utilisation des propriétés des fonctions bilinéaires, trilinéaires et n-linéaires (grandeur proportionnelle à plusieurs autres grandeurs indépendantes entre elles).

Pour résumer, la notion de champ conceptuel permet d'étudier d'une manière mieux intégrée le développement simultané et coordonné des différents concepts nécessaires à la compréhension d'un ensemble organisé de classes de problèmes, des procédures permettant de les traiter, et des sytèmes symboliques permettant de les représenter. Les hiérarchies que ces études permettent de mettre en évidence ne forment pas un ordre total mais un ordre partiel, contrairement au modèle de la théorie des stades. La notion de champ conceptuel réhabilite les contenus de connaissance, trop souvent minimisés ou effacés par l'approche structuraliste.

THEOREME EN ACTE ET CALCUL RELATIONNEL

La solution de problème est la source et le critère du savoir. C'est dans la solution de problème, ou plus généralement dans le traitement de situations-problèmes que sont élaborées les notions, et que sont abstraites les propriétés pertinentes. C'est aussi dans la solution de problème que sont éprouvées les connaissances opératoires. Le psychologue et le maître peuvent se former une image des connaissances et représentations des élèves à partir des observables dont ils disposent, c'est-à-dire des actions du sujet en situation et des témoignages symboliques que le sujet fournit de son activité: formulations verbales, dessins, schémas, écritures ...

Les différentes réponses et solutions apportées par les élèves peuvent être considérées comme engendrées par des règles de production, ou procédures. Il est méthodologiquement décisif d'identifier ces règles ou procédures. Mais on ne peut comprendre leur signification que si elles sont rapportées aux relations auxquelles elles s'appliquent. En d'autres termes, il faut les considérer comme des "théorèmes" implicites. Le concept de "théorème en acte" désigne les propriétés des relations saisies et utilisées par le sujet en situation de solution de problème, étant entendu que cela ne signifie pas qu'il est pour autant capable de les expliciter ou de les justifier.

Le concept de "calcul relationnel" désigne de son côté les compositions déductives (et les inférences) qui rendent compte de ses productions. Un exemple permettra d'éclairer ce point: plusieurs auteurs (Freudenthal, Lybeck, Vergnaud) ont relevé le fait que, dans les problèmes de type multiplicatif, les élèves utilisent plus naturellement les relations entre grandeurs de même nature que les relations entre grandeurs de nature différente; pour calculer la distance parcourue par un train rapide, en 36 minutes, sachant qu'il parcourt 40 kilomètres en 16 minutes, de nombreux élèves (entre 11 et 13 ans) font la décomposition suivante:

$$36 = (2 \text{ fois } 16) + 4 = 2 \text{ fois } 16 + 1/4 \text{ de } 16$$

donc la distance parcourue est 2 fois 40 + 1/4 40 = 80 + 10 = 90.
En procédant ainsi, les élèves appliquent le théorème

$$f(\lambda x + \lambda' x) = \lambda f(x) + \lambda' f(x)$$

C'est un "théorème en acte", car il est rarement explicité, et c'est aussi une production inventive des élèves car un tel théorème ne leur a jamais été enseigné. Ce théorème en acte est le produit direct de la conceptualisation des élèves dans le champ des structures multiplicatives. Il repose sur deux théorèmes en acte plus élémentaires

$$f(x + x') = f(x) + f(x')$$
$$f(\lambda x) = \lambda f(x)$$

et sur leur composition.
Le concept de "théorème en acte" renvoie donc à celui de "représentation calculable implicite" (ou encore de théorie implicite).

VARIABLES DIDACTIQUES ET EFFETS DIDACTIQUES

Jean Piaget a le mérite d'avoir présenté, avec la théorie de l'équilibration majorante, une théorie cohérente de l'évolution des connaissances: la connaissance passerait d'un état d'équilibre à un autre par un déséquilibre de transition au cours duquel les relations prises en compte par le sujet dans l'état antérieur seraient mises en contradiction, soit par la prise en considération de relations nouvelles, soit par une tentative nouvelle de les coordonner. Cette phase de conflit serait surmontée au cours d'une phase de réorganisation et de coordination qui aboutirait à un nouvel état d'équilibre. Si l'on applique cette théorie aux connaissances mathématiques, on est amené à considérer que les situations-problèmes présentées aux élèves constituent un levier important pour faire évoluer leurs représentations et leurs procédures. Guy Brousseau a développé à ce sujet une "théorie des situations didactiques" dans laquelle il fait jouer un rôle important à la valeur des variables numériques utilisées et à la quantité d'information à traiter. Les procédures et les notions des enfants se forment en général pour des valeurs petites et entières des variables numériques; le changement de valeur est de nature à faire échouer les procédures "locales" des élèves et à les obliger à élaborer des procédures plus puissantes. Il existe d'autres variables de situation, sur lesquelles il est possible d'agir pour produire des effets didactiques. On peut citer en premier lieu la nature de la tâche cognitive, c'est-à-dire la nature de la classe de problème et des opérations de pensée nécessaires à sa solution: par exemple dans les problèmes de type additif et pour de jeunes élèves (6 à 8 ans), les questions portant sur l'état initial d'une grandeur connaissant la transformation qu'elle a subie et son état final, sont de nature à obliger l'enfant à analyser davantage les propriétés des transformations et à découvrir la propriété d'inversion.

état initial — dépense de 4 francs — état final

exemple : $\boxed{?} \xrightarrow{(-4)} \boxed{7}$

De même l'accroissement du nombre de données à traiter et la présence de données inutiles sont de nature à rendre fonctionnelle et économique l'utilisation d'un diagramme, d'un tableau, d'une équation. Un bon exemple de l'utilisation des variables de commande didactique est fourni par le travail d'Annie Bessot et Françoise Richard, sur la combinatoire; un exemple sur les proportions sera présenté oralement.

Toutefois, le changement des variables de situation, au lieu de permettre le passage à des procédures de niveau supérieur, peut entraîner la régression à des procédures moins puissantes, mais plus fiables, ainsi que le montrent les recherches de Claude Comiti et Annie Bessot. Plusieurs autres travaux présentés à PME (Douady-Perrin, Artigue-Robinet) s'intéressent ainsi à l'effet didactique de la manipulation des variables de situation. Ces effets ne sont pas automatiques, l'état des connaissances antérieures du sujet est évidemment un élément déterminant de l'apparition ou de la non-apparition de ces effets. Les effets didactiques sont plus aisément constatables au plan des procédures de traitement qu'au plan des concepts. Pourtant on peut repérer certains exemples d'élaboration de concepts nouveaux en situation: l'exemple de la composition de transformations additives montre que les élèves parviennent à traiter certaines situations conceptuellement complexes avec des moyens plus rudimentaires, en opérant des glissements de sens adéquats; on peut alors choisir les variables de telle manière que les glissements de sens soient rendus impossibles et (espérer) provoquer ainsi l'élaboration d'un nouveau concept, celui de la composition de transformations (exemple donné oralement).

SIGNIFIANTS ET REPRESENTATIONS SYMBOLIQUES:

Les mathématiciens font un usage de plus en plus systématique, notamment dans l'enseignement, de représentations symboliques canoniques (graphiques, tableaux, diagrammes, ...). Remarquons au passage que les égalités et les équations forment également une représentation symbolique ainsi que le langage naturel lui-même. Faute de faire suffisamment la distinction entre le concept et sa représentation, c'est-à-dire entre le signifié et le signifiant, il arrive fréquemment qu'on prenne les symboles et les opérations sur ces symboles pour l'essentiel de la connaissance et de l'activité mathématiques, alors que cette connaissance et cette activité se situent principalement au plan conceptuel. Des questions importantes se posent au plan des signifiants. Par exemple, les différents systèmes de signifiants ne mettent pas également en évidence les différentes propriétés des relations mathématiques qu'elles symbolisent: l'algèbre se prête bien aux manipulations de symboleset à l'utilisation des opérations sur les nombres, les graphiques permettent mieux l'estimation des ordres de grandeur, la croissance, la décroissance, la continuité, etc. A l'inverse, ils ne se prêtent guère au calcul. Quelques recherches ont été conduites sur les diagrammes et sur les tableaux, de manière à déterminer quels aspects des relations ces représentations permettaient de mieux symboliser et de traiter (exemples donnés oralement).

L'étude des systèmes de représentation symbolique montre aussi que leur lecture et leur utilisation pose des problèmes spécifiques: par exemple le placement de données numériques sur une droite graduée, suppose des opérations de pensée beaucoup plus complexes qu'on ne l'imagine habituellement et s'avère relativement difficile pour des élèves de 10 à 13 ans (exemple donné oralement). Enfin la formulation en langage naturel de certaines relations mathématiques et de certaines propriétés soulève à elle seule un problème considérable. Colette Laborde et Michel Guillerault traitent cette question dans leur recherche. Pour ma part, je me contenterai de rappeler ce que j'ai dit tout à l'heure sur les "théorèmes en acte".

Utiliser une"propriété correcte" au cours de la solution d'un problème est essentiel. Mais cela ne fait pas une théorie mathématique explicite. Seule la formulation dans le langage naturel, et éventuellement dans un système symbolique adéquat, permet d'objectiver clairement les propriétés utilisées, de distinguer les propriétés correctes des propriétés fausses, et de leur apporter un début de validation. Une partie importante de la didactique consiste justement dans ce souci d'explicitation, qui est d'ailleurs indissociablement lié à la construction des objets mathématiques proprement dits comme ceux de nombre, d'ensemble de nombres, de fonction, de relation, d'espace, de plan, de droite, de groupe, etc.

La didactique repose donc sur deux pieds principaux:

- la construction de situations significatives pour l'élève et productrices d'effets didactiques;
- l'explicitation des relations traitées, l'analyse de leurs propriétés et la construction des objets mathématiques pertinents.

L'utilisation de systèmes symboliques spécifiquement mathématiques, ayant une syntaxe explicite, apparaît alors indispensable. Mais elle ne doit pas masquer le problème plus fondamental qui lui est sous-jacent, celui du concept.

CONTRAT DIDACTIQUE

Je ne peux pas terminer cette analyse rapide de nos cadres théoriques sans mentionner ce que Guy Brousseau a appelé le "contrat didactique", c'est-à-dire l'ensemble des attentes implicites qui règlent le fonctionnement de la classe et les rapports entre le maître et les élèves. De nombreux malentendus surgissent en effet à ce niveau, qui sont responsables d'interprétations erronées des énoncés de problèmes et des demandes du maître. La notion de contrat n'est pas une notion purement sociale, elle implique directement le contenu des connaissances. Je laisse à Guy Brousseau et à d'autres le soin d'illustrer ce concept.

ORIENTATIONS METHODOLOGIQUES

Les questions méthodologiques pourraient à elles-seules justifier un long exposé. Je m'en tiendrai ici à deux idées principales:

1. la nécessité de recourir à une diversité de méthodes;
2. la nécessité de développer l'expérimentation à l'intérieur de la classe.

L'enseignement et l'acquisition des connaissances mathématiques constituent un objet d'étude trop complexe pour qu'on puisse espérer le comprendre avec une seule approche méthodologique. Plusieurs d'entre nous ont donc fait le choix de l'aborder de plusieurs manières: entretiens individuels, expériences papier-crayon obéissant à un plan expérimental strict, expériences dans la classe, analyses de manuel, recherches sur l'histoire de l'enseignement des mathématiques, questionnaires destinés aux maîtres, etc. La finalité de ces différentes méthodes n'est pas la même.

Les entretiens individuels de type clinique et critique, tels que Piaget les a développés, demeurent à mon avis une méthode essentielle pour l'analyse des difficultés conceptuelles et pour l'analyse des procédures par lesquelles les élèves traitent une situation donnée. Ils sont indispensables pour éprouver le solidité des conceptions des élèves (grâce à l'utilisation de la contradiction) et pour analyser l'évolution des procédures et des conceptions en situation. Ils permettent de mieux faire la part entre les conduites vraiment significatives d'une conception, et les artefacts ou autres réponses anecdotiques.

Il s'agit cependant d'une méthode coûteuse en temps, et il serait déraisonnable de se priver des expériences papier-crayon, qui ont le mérite de toucher de larges échantillons, et de permettre ainsi, lorsqu'elles ont été planifiées dans ce but, de faire des comparaisons multiples qui seraient impossibles avec les entretiens individuels.

Par exemple, Graciela Ricco, André Rouchier et moi-même avons conduit 80 entretiens individuels pour étudier la compréhension des propriétés de trilinéarité du volume, à travers une série d'items où les gestes, les dessins et les silences des sujets étaient importants. Jamais nous n'aurions pu obtenir les mêmes informations avec une autre méthode. Inversement, nous avons conduit une étude papier-crayon sur la règle de trois, qui permet un grand nombre de comparaisons en fonction des valeurs numériques choisies: cette étude aurait été dispendieuse avec la méthode des entretiens.

Que représente l'expérimentation en classe par rapport à ces méthodes que je viens de rappeler et que tout le monde connaît?

L'expérimentation en classe, comme toute expérimentation, n'a de sens que si on la conduit avec une problématique claire en tête, c'est-à-dire si l'on a fait un choix réfléchi du thème et des aspects étudiés. Il est vain de se livrer à une observation du travail en classe, si l'on a pas élaboré soigneusement les situations et les questions proposées aux élèves, et si l'on n'a pas cherché à expliciter, aussi clairement que possible, les objectifs didactiques qu'on se donne, les hypothèses qu'on fait sur ce qui va se passer: par exemple sur le volume en classe de cinquième (élèves de 12 à 13 ans environ) nous avons construit une séquence didactique que présentera André Rouchier: pour chaque séance, nous avons réfléchi longuement à la situation présentée, aux objectifs que nous voulions atteindre, aux conduites que nous attendions et aux effets que nous espérions produire. Ni le choix du matériel, ni le choix des valeurs des variables, ni l'enchaînement des questions, ni leur formulation ne peuvent être laissés au hasard. C'est à cette condition seulement que nos observations prennent un sens et sont suffisamment précises. Sans cette condition, l'observation demeure vague, peu fiable et peu répétable.

L'enjeu scientifique est en effet important: est-il possible, au niveau d'une séquence didactique relativement longue et complexe, et au niveau de toute une classe, d'observer des régularités dans les événements qui se produisent, régularités interprétables par l'analyse des conditions qui leur ont donné naissance, pouvant comme telles être considérées comme des faits didactiques. Lorsque les psychologues ont développé la technique des entretiens individuels cliniques, ils ont été confrontés à un problème analogue, et ce n'est qu'avec de multiples précautions qu'ils ont pu mettre en évidence des faits psychologiques incontestables et répétables, au moins à un certain niveau de lecture des observations.

On peut espérer qu'il en ira de même dans la recherche en didactique et qu'il sera possible d'établir des faits didactiques, c'est-à-dire des faits concernant la transmission et l'appropriation du savoir par l'action finalisée de l'école. Le critère de la répétabilité est un critère exigeant. Il est impossible de le satisfaire à la lettre, mais il est peut-être moins malaisé qu'on le croit de s'en approcher, à un certain niveau d'interprétation des événements. En effet, lorsqu'on répète la même séquence didactique avec plusieurs classes d'élèves, on constate des régularités très importantes, notamment au niveau des distributions de conduites , et les variations prennent elles-mêmes un sens par rapport à ces régularités. La répétition permet en tout cas d'améliorer la formulation des objectifs et des hypothèses et de remettre en chantier les propositions didactiques qui ont fait l'objet de l'expérience.

L'expérimentation en classe n'est pas pour autant la voie royale de la recherche, d'une part parce qu'elle ne permet pas, même avec de bons moyens d'enregistrement, d'analyser dans le détail tous les processus en jeu, d'autre part parce qu'elle est d'autant meilleure qu'elle peut s'appuyer sur les résultats obtenus par d'autres méthodes (entretiens individuels, expériences planifiées). En retour elle permet de déceler des phénomènes qu'il serait intéressant de regarder avec la loupe des entretiens individuels. En tout cas, on ne voit pas comment la recherche en didactique pourrait faire l'économie de l'expérimentation en classe.

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Research on Mathematical Learning and Thinking in the United States[1]

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Any survey of the educational research scene in the United States is bound to be not only limited, but also rapidly outdated. The scene is complex, and it is volatile. These observations are only slightly less true if one's view is restricted to research on mathematical learning and thinking. Before one movement has begun to fade, another comes along, and although there are important ways in which the scene remains relatively constant over time, US research on mathematical learning and thinking is affected by the swirling currents in the surrounding seas of educational research. The metaphor of the pendulum is often invoked to characterize change in mathematics education in the United States, but I find that metaphors relating to weather and the ocean capture more of its rhythm and unpredictability.

If we look, for example, at the learning theories that have been used as bases for mathematics teaching, we see that the tides of theory can change quickly. When Howard Fehr (1953) surveyed the then-prominent learning theories of conditioning, connectionism, and gestalt psychology to identify implications for classroom practice, he did not note the wave of cognitive psychology that was about to sweep American educational research. When Lee Shulman (1970) used the issue of discovery learning to contrast the implications for mathematics teaching of Bruner's cognitive developmentalism with Gagné's neobehavioralism and Ausubel's cognitive theory of meaningful verbal learning, he did not anticipate that within a few years not only would discovery learning be largely abandoned as an issue in American mathematics educational research, but Ausubel and Bruner would no longer be active in educational research and Gagné would have recanted much of his neobehavioralist position. At present, we are enjoying the brisk breezes of information-processing psychology, but who can say what the weather is just over the horizon?

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As one measure of the difficulty I have in seeing what is happening in the mathematics education research scene, let me cite an observation I made eight years ago concerning the annual growth in the number of research studies (Kilpatrick, 1973). I noted that there had been a jump of 44 percent in one year, from 1970 to 1971, in the number of dissertations dealing with research related to school mathematics listed in the annual survey of the Journal for Research in Mathematics Education. A look at the annual surveys for the years from 1970 to 1980, however, shows how much of an aberration that jump was. The rate of growth in the number of dissertations cited has been much more modest over the decade, and there are some signs that it may be slowing down. Somewhat more steady has been the growth in the number of journal articles published, and there has been a marked increase in the number of journals that publish articles on mathematics education research. I have not looked closely at the data, but my impression is that the greatest growth over the past few years has been in the number of authors and journals from outside the United States.

My record as an analyst, then, is not perfect. Furthermore, one can argue that a native of the culture is the wrong person to ask for an insightful picture of its condition. It is with some reservations, therefore, that I offer the remarks that follow. I shall first attempt to identify some current problems in US research in mathematics education that arise primarily from the context in which it is embedded. Much of our research has been heavily criticized for ignoring this context, so I will make sure it gets appropriate attention by treating it first. Then I shall attempt to sketch some theoretical and methodological issues in US research on mathematical learning and thinking. Finally, I shall attempt to make the discussion more concrete and focused by examining how some of these problems and issues are reflected in US research on a topic I am especially familiar with--mathematical problem solving. It goes without saying that all of these views are my own; I have no idea how widely they might be shared either within the culture or outside it.

Current Problems in US Research in Mathematics Education

Problems of Climate and Morale

The climate for research in mathematics education in the US today is not an especially healthy one. Much of the torrent of research that appeared in the 1960s and 1970s was powered by federal dollars appropriated to improve the mathematics curriculum and the teaching of mathematics--although it should be quickly added that only a tiny fraction of those dollars actually supported the research. That stream of dollars has all but dried up. Americans increasingly entrusted educational affairs to the fedaral government in the past three decades principally because of their concern about manpower needs (Spring, 1976). That tide seems to have turned, however. Although the concern remains, people have lost faith in the government's ability to meet educational needs through its policies and programs. The widely held belief that educational research efforts have been mostly ineffective suggests that, even if more money were to become available to improve American education, little of it would likely be earmarked for research. Education is turning to the states and to private enterprise for increased funding, but these agencies have a meager record of supporting research in mathematics education, and I see little likelihood that this will change soon.

One might take heart from Heinrich Bauersfeld's (1979) observation that

> it is in the periods of deep economic depression . . . that the creative framework and fundamental research are prepared which are to build the substance of the following period of ascension and flower. (p. 199)

Perhaps, as the frantic data gathering slows down, more penetrating and scholarly analyses will appear in the literature of US mathematics education. Perhaps. But we face a more serious problem than economic depression; we in the United States appear to have lost much of our faith in the public school's power to transform our people and our society. A devaluation of education as an institution and as a profession pervades American life, demoralizing teachers and researchers alike. The duration, depth, and ultimate impact of this depression of the spirit are impossible to foresee.

Problems of Identity and Status

Bauersfeld (1979) also noted that the mathematics educator must "develop his own self-concept" (P. 210). This is happening in the United States to some extent, but the community of researchers in mathematics education lacks the coherence and identity a true community ought to have. It is not simply that a comprehensive professional organization is lacking; the Special Interest Group for Research in Mathematics Education of the American Educational Research Association, together with the North American branch of PME, fill much of the need for organization. And it is not simply that doctoral dissertations that are the candidate's first and last venture into research continue to flood the terrain--although that is a serious problem. The root problem runs deeper and touches on matters of professional identity and status. It is perhaps not surprising that, with some notable exceptions, US mathematicians view research in mathematics education with a disinterest that borders on disdain: the status differential between the mathematician and the educationist is deeply embedded in our culture. Harder to explain is the (largely covert) disparagement of each other's work that one sees between researchers in mathematics education and researchers in educational psychology in the United States. Despite much borrowing across the fence between the two fields, these neighbors--again with some exceptions on both sides--strongly resent encroachments on their territory. Relations between researchers in mathematics education and mathematics teachers are somewhat more cordial, probably because neither poses much of a threat professionally to the other. All of this territoriality on the part of US researchers in mathematics education might be more productive of a group identity if the territory itself were more clearly marked out. Uncertainty about its boundaries, however, is pervasive.

Problems of Purpose and Effectiveness

Some of the uncertainty as to what constitutes research in mathematics education in the United States arises from accusations that it has been ineffective in changing school practice. The consensus on this point seems to be that too many researchers have been studying the wrong things in the wrong ways. Researchers are (perhaps overly) disillusioned with traditional methods of educational research, but they are uncertain about how to use newer methods appropriately and effectively. Many people are convinced that the wrong research questions have been asked, but few examples of the right questions have been proposed. Lurking below the surface is the issue of how research should affect educational practice. Outside of mathematics education one hears the argument that, on the one hand, the best educational research is both pure and applied (Greeno, 1978) and, on the other hand, the researcher and the practitioner ought to expect little from each other (Phillips, 1980). Inside US mathematics education, one hears mostly silence on this issue.

Theoretical Issues in US Research on Mathematical Learning and Thinking

The preceding gloomy litany of problems provides a backdrop against which to view current theoretical issues, as I see them, in US research on mathematical learning and thinking. To get a better sense of how US researchers are dealing with theory, I examined the 38 articles in the ten issues of the Journal for Research in Mathematics Education from July 1979 to May 1981. Of these articles, 35 had authors with US affiliations only. I looked at each article to see if an attempt had been made to link the question under investigation to some explicit theoretical context. For 20 of the articles, I could find no such attempt. Six of the remaining articles dealt with aptitude-treatment interactions, and although the authors seldom specified the theoretical bases for their activities, I gave them credit for some contact with theory. Of the remaining nine articles, three made more than casual use of Piaget's genetic epistemology, two were linked with expectation theory, and one concerned each of the following: causal attribution theory, a theory of attention, an application of graph theory to the representation of cognitive structure, and information-processing theory. I may have been too harsh in some of my judgments, but I conclude from this and other observations that a lack of attention to theory is characteristic of US research in this field. Further, when theoretical constructs and contexts are used, they are not "home grown"--they are today, as they have been for years, borrowed from outside mathematics education. As Donald Sanders (1981) has noted:

> [We have a] tendency to approach educating through constructs rooted in psychology or the social sciences rather than through theories or constructs fitting to phenomena as they appear in educational settings. . . . Educational research rooted in the theories and paradigms of related disciplines may advance those disciplines, but it does not necessarily advance scientific knowledge of the process of educating. (pp. 9-10)

One of our greatest needs in research on mathematical learning and thinking is for conceptual, theory-building analyses of the constructs and assumptions we are using. Such analyses must, of course, be informed by empirical data--and, paradoxically, "despite rampant empiricism . . . there is very little trustworthy data representing the facts of the educating process" (Sanders, 1981, p. 9). Another need is to shift more of our attention to the study of mathematics learning and thinking as they occur in school. We must take full account in our research of the multiple contexts in which both learning and thinking occur. Each is embedded in interacting systems of the pupil's cognitions, the subject matter, and the social setting. We have tended to concentrate on at most one of these systems, and we have neglected interactions within the system, not to mention interactions between systems. For example, the study of concept learning has too frequently been pursued by US researchers as though the learning of a mathematical concept were uninfluenced by the learner's knowledge, the existence of related mathematical concepts, or the teacher's beliefs about the concept. To design and conduct studies that can handle the complexities of these multiple contexts is perhaps the biggest challenge we face. A final need is to devote some attention to scholarly inquiry into and reflection on our own research activities. Scriven (1980) terms this "self-referent research," and at first it may seem just another gimmick to find something else to do research on. But just as metacognition--cognition about one's own cognitions--is indispensable for intellectual growth, so some metaresearch efforts are required for the growth of our field.

Methodological Issues in US Research on Mathematical Learning and Thinking

The key methodological issue, as I see it, is how to take full account of the complexity of mathematical learning and thinking by adopting and adapting new methods of investigation without at the same time compromising our standards of quality in conducting and reporting research. US researchers are actively exploring approaches such as the teaching experiment (Kantowski, 1976), the case study (Stake, 1978), and the ethnographic study (Rist, 1980). Each approach holds promise, but no approach will suffice by itself. Some researchers are advocating more "artistic" approaches (Eisner, 1981), but it is an open question whether such approaches can be pursued successfully by people who have been trained as scientists and not as artists. The oldest refrain in research in mathematics education is the call for replication studies. The new refrain ought to be a call for convergent lines of investigation in which studies explore a question from a variety of perspectives using a variety of methods.

US researchers are presently rather disenchanted with quantitative methods of data analysis--although one might not necessarily conclude that from reading the articles in the Journal for Research in Mathematics Education. There is some danger, I think, that the reaction may be too extreme, and the quantitative baby may go out with the statistical testing bath water. Rather than abandoning quantitative methods in favor of qualitative ones, our efforts should be directed towards enriching both. In particular, new techniques of exploratory data analysis and presentation should be explored by researchers in mathematics education, and they should consider techniques for the reanalysis of data and the meta-analysis of studies.

Also, the development of instruments to assess mathematical learning and thinking ought to be given priority as a valid, vital research enterprise. We have been too prone to investigate those constructs for which we have instruments at hand rather than taking the time and effort to develop new instruments for assessing our constructs.

Research on Problem Solving in Mathematics

The designation of "problem solving" as the catchword of the 1980s in US mathematics education ought to gladden researchers concerned with mathematical problem solving, and one can expect that their ranks will swell in the next

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few years, but the problems of morale and identity touch them, too. The profession seems to have convinced itself that problem solving is important in school mathematics; the public, however, may not understand, or accept, this argument in the same way. Claims that research is needed to understand how children solve mathematical problems and how to teach them to solve such problems are likely to fall on uninterested ears. Some US researchers on problem solving in mathematics have formed informal networks of association that have been developed through working groups of the Georgia Center for the Study of Learning and Teaching Mathematics and through the National Science Foundation's meetings of project directors. These networks are fragile, however. I am not sure they can withstand the impending heavy weather of further reductions in federal funding.

Researchers on problem solving are as guilty as anyone in US mathematics education of failing to develop clear theoretical rationales to support their work. The name of Polya is invoked as though that absolved the writer of any responsibility to provide a scholarly analysis of problem solving. The term "process" is used as though its meaning in the context of mathematical problem solving had received a clear exegesis. Mathematics educators should critically examine the possible contributions of information-processing psychology to our theory-building work--to see whether, as Resnick and Ford (1981) assert,

> for the first time, psychology has a language and a body of experimental methods that is simultaneously addressing both the skills involved in performance and the nature of the comprehension underlying that performance. (p. 197)

The stress that information-processing psychology gives to the role of memory is reflected in recent work such as that of Ed Silver (1981), and researchers such as Alan Schoenfeld (1981) are attempting to adapt the information-processing notion of "executive decisions" to the analysis of mathematical problem solving. We still lack, however, those "home grown" adaptations of the information-processing metaphor that would illuminate the specifically mathematical characteristics of mathematical problem solving.

With regard to methodology, I would respectfully suggest that the proliferation of schemes for analyzing "thinking aloud" protocols is not getting us very far. It would be good to see more attempts by researchers in mathematics education to check results obtained from protocol analysis against results obtained by other methods, such as reaction-time studies. More studies ought to employ multiple methodologies--in the spirit, if not on the scale, of Krutetskii's (1976) work. (Some examples of recent studies in which multiple methodologies were used can be found in Harvey & Romberg, 1980.) Serious attention should be given to the tasks used in problem-solving research. Some recent work has been done on this issue (Goldin & McClintock, 1979), but better instruments for assessment are still needed.

I end this cursory survey on a note that I wish could resonate through the ranks of mathematics educators who do research in the United States--and perhaps it is needed in other countries, too. Research on mathematical problem solving is beginning to move out into the classroom, and the work of such people as Mary Grace Kantowski at Florida, Perry Lanier at Michigan State, Dick Lesh at Northwestern, and Nick Branca at San Diego State bears witness to the wisdom of this move. But we still need to reconceptualize our approaches so that mathematics teachers can become full partners in the research enterprise. As observers such as Sanders (1981) have pointed out, in no profession other than education does one find such a sharp differentiation between the community of researchers and the community of practitioners. The time seems ripe, if we are to move ahead confidently and productively, for research in mathematics education to develop its self-concept as an endeavor demanding stronger links with practice as well as better scholarship.

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Towards a Research Consensus in Some Problem Areas in the Learning and Teaching of Mathematics

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A significant turning point in the history of research in any araa occurs when from the chaos of competing ideas about a problem area, a single paradigm emerges which implicitly defines for practitioners the legitimate problems and methods of research. I believe that current work on several problem areas within the learning and teaching of mathematics should be viewed as foreshadowing the emergence of a firm research consensus in these areas.

TOWARDS NORMAL SCIENCE

To build this argument, I follow Thomas Kuhn's description of the "route of normal science." In his now classic treatise on the growth of science, The Structure of Scientific Revolutions (1979), Kuhn argues that the road to a research consenus in any area is arduous. In the absence of a paradigm or set of organizing principles, all facts that could possibly pertain to a problem area are likely to seem equally relevant. As a result, early fact-gathering is a nearly random activity. Furthermore, in the absence of a reason for seeking some particular form of instruction, early fact-gathering is usually restricted to the wealth of data that lie ready at hand. Thus, facts accessible to observation and experiment are pooled together with data retrievable from reports of classroom teaching, curriculum development, or program evaluation.

Although early fact-collecting has been essential to the origin of many significant sciences, one somehow hesitates to call the resulting literature scientific. Research in mathematics education during the decade of the 60s was largely that type of random fact-gathering. For example, in 1969, for the Review of Educational Research issue on science and mathematics education, I argued that research at that time could "be characterized as large in quantity, poor but improving in quality, and diverse" (Romberg, 1969, p. 473). The diverse problems being studied at that time made it difficult to classify studies even into reasonable problem areas. Hence, the copious literature simply lacked an implicit body of intertwined theoretical and methodological beliefs that permitted easy selection, evaluation, and criticism. Ralph Tyler echoed this argument in his summary for the National Conference on Needed Research in Mathematics Education.

> I conclude that we seem to have reached a consensus on a broad definition for research in mathematics education and upon the value of such research for the practical enterprises of teaching and learning in mathematics. There still remain two frequently voiced criticims: (1) current research is not sufficiently discriminating in selecting problems that are significant; and (2) current research seems to be scattered in the sense that it is difficulty to put together the various investigations and their findings in a way that provides the cumulative development of knowledge, thought to be characteristic of the natural sciences (Tyler, 1969, p. 135).

As the decade drew to a close, salesmanship gave way to questioning, and in some cases, to careful inquiry. We began to realize that learning and teaching in schools is complicated. The changes in content alone (called the "modern math" revolution) were not sufficient to produce drastic changes in mathematical learning.

The next decade might be characterized as a period of confrontation, confusion, and reflection. For example, by 1975, the Subcommittee on Research at the National Institute of Education sponsored Euclid Conference on Basic Mathematics Skills and Learning argued that we had now gone beyond fact-gathering.

> We agree with the critics that past research in this area (mathematics education) has been inadequate. Past studies can be characterized as a plethora of piecemeal studies rather than sets of studies reflecting scholarly chains of inquiry. Too many studies have been based on an inadequate conceptualization of the problem being investigated, and have employed poor instrumentation and inappropriate methodology. We feel that these faults of past and current research are the typical characteristics of emerging fields of inquiry. On the other hand, it is clear that we now know more about the teaching and learning of mathematics than we did some fifteen years ago before there was substantial federal support for educational research. In particular, we have eliminated some options which seemed at one time to be viable but proved to be unproductive and we have developed a much more sophisticated research methodology and have identified some potentially promising directions for research. For example, we know not to rely solely on simplistic frameworks such as behavior modification of discovery learning to solve our problems, not to use standardized performance tests as sensitive dependent variables, and not to rely on quasi-experimental designs from agriculture as canons of research methodology. In summary, we feel that a promising start has now been made both in terms of research completed and in terms of improvements in research methods (Romberg et al., 1975, pp. 2-3).

The literature of the past decade was filled with arguments such as: behaviorism vs. constructivism, quantitative vs. qualitative methods, advocates of open schooling vs. traditional, proponents of activity instruction vs. seatwork, and so forth. The claims, counterclaims, and arguments of researchers must have been confusing to teachers, administrator and graduate students. But, it really should be no surprise that, in the early stages of the development of any science, scholars confronting portions of the same range of phenomena arrive at different descriptions and interpretations. However, during such a period of confusion, there is envitably reflection. Some scholars are able to sift through the morass and through reflection piece together concepts and propositions which encompass and clarify the conflicts. What happens then is a group previously interested in the general study of a field, like mathematical learning and teaching is transformed into a scientific group committed to a specific problem area like adolescent learning of rational numbers. By the end of the past decade, I am suggesting that for some problem areas, within mathematics education, consensus is beginning to emerge, as in counting and early learning of numbers, learning of rational numbers, and verbal problem solving.

Kuhn argues that such consensus has two essential characteristics. First, the synthesis by bringing clarity to a confused area is sufficiently unprecedented to attract a group of adherents. Simultaneously, the synthesis is open-ended, leaving all sorts of problems for the redefined group of practitioners to work on.

Elsewhere, I have argued that "the key step in (the) evolutionary sequence from myth and tradition to theory is model building" (Romberg, 1981, p. 186). The purpose of building a model is to unpack the broad assertions about a problem in order to elucidate the key variables and the relationships between them. Going from macro-assertions to micro-analysis is the critical step.

The development of a model starts in some empirical situation that presents a "problem" for which an "answer" can be very misleading. One must recognize that real situations rarely appear well-defined and are often embedded in an environment that makes it hard to obtain a clear statement of the situation. Formulating the problem involves specifying the assumptions, concepts, and principles one believes are operating in the real situation. Such specification must of course be selective by its bias, and most often deals with only a small part of the larger problem. This simplification or idealization is important since the general problem is usually exceedingly complex and involves many processes. Some features of reality will appear significant and many irrelevant. A model's validity rests on its ability to represent the situation initially described. What I am claiming is that for some problem areas in the learning and teaching of mathematics we are now in the model building stage.

In summation, I believe, the "route to normal science" in any area often can be viewed in terms of six stages:

Stage 1: fact-collecting. Here random studies are carried out with little focus and no sense of order.

Stage 2: confrontation, confusion, and reflection. That is a period of sorting out facts, deciding on problems, arguing about well variables, etc.

Stage 3: model building. This is a time for organizing ideas about well identified variables, etc. (The current stage addressed in this paper.)

Although not relevent to this paper Kuhn suggests there are at least three other stages that follow;

Stage 4: paradigm selection.

Stage 5: normal science.

Stage 6: scientific revolutions.

Also, one essential characteristic marking the distinction between the notion of model and the classical notion of theory is that the point of a model often lies more in whether it illuminates the nature of a phenomenon than in whether it is true. There is nothing outrageous in the fact that models that generate very different propositions about the same phenomenon can coexist without invalidating each other.

Finally, whereas a theory is conceived of as having a certain minimum of generality, instances can be quoted of mathematical models that explain only specific facts. This notion of model contravenes the postulate that science deals only in generalities. We can go even further and say that recent progress in the social sciences is partly connected with the fact that it has been considered worthwhile to analyze things that, without necessarily being unique, are specific. Admittedly, this kind of model usually includes some general propositions, but these propositions by themselves are inadequate for explaining an observed reality. In summary, if a model is falsifiable, uniquely fits the phenomena under study, and has some generality (often called a "fitting" model), it is a theory. Otherwise, it is a "guiding" model. It is important to understand that this characterization of modeling is utterly at variance with the epistemological notions offered by classical physics, to which the social and human sciences clung slavishly throughout the whole of the nineteenth century and part of the twentieth. Because having a "fitting" model implies properties such uniqueness, generality, and verifiability, at present it is unsuitable for describing the results of attempted formalizations applied to the analysis of mathematics teaching and learning. Only by posing "guiding models" will we ever be able to build a theory of mathematical education.

AN EXAMPLE: TEACHING MATHEMATICS IN ELEMENTARY CLASSROOMS

Several problem areas in mathematics education could have been chosen to illustrate the trend toward research consensus. For example, Carpenter, Blume, Hiebert, Anick, and Pimm (1981) have just completed an extensive review of early number research and I (Romberg, in press) have argued that consensus is nearing in that area. Similarly, Swafford (1981) has traced the evolution of assessing attitudes toward mathematics and model building has begun (see Nimier, 1980).

For this paper, I have selected classroom instruction, in particular, what teachers do in elementary schools. Space does not permit a detailed consideration of all the issues associated with this development but let me pose the problem and trace the trend toward consensus in this area.

To me the problem is self-evident. Teaching little children is hard work; but the teaching of mathematics in most primary school classrooms after modern mathematics programs were introduced was not just hard it was poorly done. The writing of such schooling critics as Charles Silberman, Crisis in the Classroom (1970); John Holt, How Children Fail (1964); Carl Bereiter, Why Is Teaching Mathematics So Awful?; H. W. Sobel, "The anacronistic practices . . ." (1969); or Morris Kline Why Johnny Can't Add (1973) to name a few are filled with examples of bad mathematics teaching in primary school. Perhaps it is impossible to teach the initial ideas of mathematics well, given the state of our knowledge about mathematics and how children learn. And, probably the teaching of mathematics in primary school has always been poor.

But, it has become a problem of research interest as a result of the "modern mathematics" movement. The intentions of the curriculum reform movement were honorable, but in retrospect, the reformers in attempting to organize mathematics, failed to see the complexity of the pedagogical problems they faced. Because proposed curriculum activities were not often understood by teachers, the resulting instruction too often became chaotic, stultifying, and at best dull and non-motivating.

During the 1960's researchers interested in this problem area found little in the existing research to help clarify the problem. The publication of the Handbook of Research on Teaching (Gage, 1963) helped scholars to see the complexity of the problem. Reviews during the era, such as those by Davis (1967) and Fey (1969), indicated the lack of consensus on how to study the problem of classroom teaching, what was involved, etc.

During the 1970's, two lines of research, best summarized by Bellack (1978 and 1981), became prominent. In the first approach, rooted in behavioral psychology and called "process--product research" (Dunkin & Biddle, 1974), observers count frequencies of classroom behaviors (processes). Then these frequencies are correlated with student performance (product). Gage's recent book, The Scientific Basis of the Art of Teaching (1979), summarizes this line of inquiry. Recent syntheses of this research suggest that for mathematics in the elementary grades, the classroom with the following characteristics leads to increased academic performance (Rosenshine, 1976; Rosenshine & Berliner, 1976; Medley, 1977; Brophy, 1979):

- The teacher keeps the students on academic tasks, and the content coverage is extensive.
- The teacher and workbook questions are highly structured, and elicit a relatively high rate of correct answers from students.
- The teachers and materials provide immediate, academically-oriented feedback, praising correct responses and exploring incorrect ones.
- Instruction is provided to the whole class or to small groups.
- Teachers monitor student performance during recitation sessions, and provide individualized feedback to students.

The second approach, rooted in sociology and called "interpretive research" (Bellack, 1981), involves observers describing the flow of classroom events as perceived and interpreted by teachers and students. Studies of classrooms, such as: Sharp and Green (1975), Bowles and Gintis (1976), Bossert (1979), Lundgren and Pettersson (1979) and recently Popkewitz, Tabachnick, and Wehlage (1981) picture schools with characteristics quite different from the first approach (Romberg, 1980).

- Schooling is a collective experience for children.
- Time is the major control mechanism of schools.
- The students at a particular age are assumed to be more similar to each other than they are different.
- Instruction within a time segment involves children working on a lesson which addresses competition, evaluation, order and control.
- Overt knowledge and skills to be transmitted to the children are expressed in terms of cognitive terms rather than social vocational development.

These five characteristics are traditional elements which make an American school a school. The job for teachers then is to assign lessons to a class of students, start and stop the lessons according to some schedule, explain the rules and procedures of the lesson, judge the action of the students during the lesson and maintain order and control throughout. For students the job is to be active participants in each lesson, attend to the explanation of rules and procedures, work independently on tasks, and try to be successful.

Also, during the 70's a lot of other activities not directly part of either main approach contributed to the problem area such as: program evaluations in mathematics classrooms (e.g., Romberg, 1976); retrospective accounts of curriculum implementation (e.g., Shulman & Tamir, 1978); studies of organizational decision making (e.g., Anglin, 1976); and analytic critiques of research (e.g., Doyle, 1978).

Today, many of the ideas from these approaches and independent activities are being brought together in "time-on-task" models. Borg (1980) has traced the historical background of such models. One such model has been used in the Beginning Teacher Evaluation Study (BTES) (Fisher et al., 1980) and more recently in the IGE Evaluation Project (Romberg, 1976), Coordinated Studies in Mathematics (Romberg, Small, & Carnahan, 1979), and the Sandy Bay Study (Romberg & Collis, 1980). Excellent reviews of this approach have been written by Westbury (1979), and Anderson (1980). In fact, the steps associated with an instructional model based on this had been outlined recently by Good (1981).

The past 20 years of research on classroom teaching in mathematics has followed the stages outlined above. Early research dealt with describing the phenomena of classrooms and how they operate. This was followed by considerable controversy, reflection, alternate approaches, etc. Now I believe there is emerging a consensus associated with models of how time is spent in classrooms.

CHARACTERISTICS OF TRANSITION TO MODEL BUILDING

To summarize this argument about the route to normal science, I would like to emphasize three characteristics of this evolution. The first, is the shift from "macro-analysis" to "micro-analysis." Model building includes a very careful micro-analysis of the processes that are involved in a particular problem area. Studies of "good" and "bad" teaching give way to detailed analysis of specific aspects of teaching (decision making, planning, instruction, etc.).

The second characteristic is a discipline shift. In this case from mathematics to psychology and sociology. The mathematics to be taught and the way it is to be organized as the areas of principal concern has been replaced by behaviors and intentions. I must admit this shift in some cases has been over done as Buchman (1980) has recently argued.

The third characteristic is the shift from individual research to collaborative programs. Lakatos (1978) has outlined the importance of chains of research which link to form a larger conceptual framework about a particular phenomena. Problems such as the study of classroom learning are complex and go beyond the capabilities of an individual researcher to investigate. Collaborative teams with members having different conceptual backgrounds (i.e., mathematicians, sociologists, psychologists, educators, teacher trainers, etc. working together) are needed.

CONCLUSION

From the time when I entered the educational research arena (nearly 25 years ago) what constituted a reasonable research agenda and how one went about doing research has changed. I believe we are near a significant turning point in the history of research in mathematics education. We have gone from a variety of atheoretic, fact collecting studies to a period of model building. We have learned to appreciate the contributions that other scholars (particularly psychologists and sociologists) can make. This is an international trend which I believe is already producing important research for the area of the teaching and learning of mathematics.

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WHAT IS A GOOD ENVIRONMENT FOR THE INTELLIGENT LEARNING OF MATHEMATICS? DO SCHOOLS PROVIDE IT? CAN THEY?

Richard R. Skemp, University of Warwick.

A ICME IV (1980) on a ébauché un nouveau modèle d'intelligence, tout en montrant le rapport avec le travail déjà fait sur la comprehension des mathématiques. On a fait ressortir la fonction adaptive de l'intelligence vis-à-vis de l'environnement physique et social. On a présenté les mathématiques comme un exemple puissant, et utile à plusiers points de vue, de cette activite de l'intelligence humaine qui contribue puissamment à notre maitrise de l'environnement physique, et qui nous rend capables de coopérer avec les autres. Elle est aussi un exemple de notre puissance creatrice. C'est surtout pour ces raisons-là que les mathématiques sont si importantes dans le monde d'aujourd'hui.

Sur plusieurs plans, l'environnement des écoles diffère de celui du monde de travail, où les mathematiques sont exploitées d'une façon pratique. En particulier l'école ne fournit à l'elève aucunne des trois situations majeures dans lesquelles les mathématiques peuvent être aperçues comme une source de pouvoir et de devertissement.

Un caracteristique important de l'intelligence, c'est qu'elle nous donne le pouvoir de construire (c'est a dire de developper et de mettre à l'épreuve) des structures conceptuelles qui, a leur tour, nous permettent d'inventer des strategies ou des méthodes pour exécuter une grande variété de tâches. On a distingué trois façons de developper, et trois façons de mettre à l'epreuve. Mais de ces six façons, deux seulement sont employées dans les méthodes conventionelles d'enseigner les mathematiques.

Puisqu'on doit toujours s'adapter à un environnement, nous devons nous demander si l'adaptation à l'environnement d'une école aide aux enfants d'apprendre les mathematiques. Je reconnais que les écoles diffèrent beaucoup entre elles, mais j'arrive, à contre-coeur, à la conclusion que beaucoup d'écoles ne constituent pas un bon environnement pour apprendre d'une façon intelligente les mathématiques.

Pourraient-elles le faire? Oui, si elles le désiraient vraiment. Mail il ne suffirait pas de ne changer que le programme scolaire.

The present paper is a direct sequel to the one I presented last year at ICME IV (Skemp, 1980). In that paper, I described how I saw mathematics as one of the most important examples of the functioning of human intelligence, by which (both in general and in this particular case) we are better able to do three things:

(i) to achieve goals in the physical world

(ii) to co-operate with our fellow beings

(iii) to create.

At the end I wrote: "I've been describing the way mathematics is for people like air navigators, scientists, technologists, bankers, industrialists, creative mathematicians. For many children, this is not the way it is. They see it variously as a job to be done, a threat to be averted, a complicated exercise for fulfilling teachers' expectations, or (in Erlwanger's memorable words) as 'a set of rules for making arcane marks on paper'." It is this contrast that I want to explore further today: between mathematics as a source of power for those who use it successfully in the adult world, and mathematics as at best a chore and at worst a source of failure as it is for so many in the world of school. Mathematics is indeed one of the most successful products of human intelligence, cumulatively over many generations. But what has been coming through to me ever more strongly since writing the ICME paper, is that the uses for whose achievement mathematical schemas are so powerful a source of help find little or no counterpart in the child's school situation. It's not just that the uses are different: it is that the kinds of use are different. Some good examples of these differences are given by Davis (1980) in his thoughtful essay "The Possibly Elusive Content that needs to be Learned", the whole of which is highly relevant to the problems on which I shall be focussing today.

In my talk today I shall be taking a wide-angle view, focussing on global features rather than detail, with situational influences and implied goals rather than the overt and explicit contents of teaching, important though the latter are also. My justification for this is that details may change their meaning in different contexts, and the most well-intentioned and well-devised teaching can fail to achieve its goal unless account is taken also of the wider context in which this is taking place, and especially of the reality of the situation as it is construed by the learner. To adapt an aphorism I heard many years ago: "What you are doing shouts so loud, I cannot hear what you say."

A human child is at the most learning age of the most learning animal that this earth has yet produced. Yet daily mathematical instruction for about 36 weeks a year for about 10 years produces the truly dismal kind of result of which we are repeatedly made aware by surveys in my own country (and the story seems to be the same, with minor variations, elsewhere): namely that children have learnt to perform routine processes well, but are bad at adapting these to the requirements of new situations. Adaptability is a key feature of intelligence, so the inference seems clear to me. Lack of adaptability suggests strongly that the mathematical learning which has this as its outcome is not making full use of children's intelligence. Mathematics is not being taught in ways which bring about intelligent learning, or it is not being taught in an environment favourable to intelligent learning, or both.

As conceived in my own model (Skemp, 1979), intelligent learning consists of
(i) the construction of appropriate schemas;
(ii) devising from these particular plans for particular tasks. These may include routine plans (algorithms) for frequently-encountered tasks.

When the tasks are non-routine, (ii) becomes problem-solving.
The first of these may be called "knowing-that", and the second and third "knowing-how". We also need "being able" - efficiency at putting these plans (methods) into action accurately and quickly. These are often called "skills", but for me skill implies adaptability, so I would only accept this term if conceptual back-up is also available if needed. It is the parent schemas from which new plans can be derived when necessary which are the sources of adaptability. So we need now to think about the processes by which these schemas come into being.

My own position here is a constructivist one, that conceptual knowledge cannot be transmitted directly. Only the learner himself can construcr these schemas within his own mind.

Teaching is an intervention in this invisible activity. A learner can be greatly helped by a teacher who has a good mental model of the processes by which the learner constructs his schemas. But if this is not the case, I can see no reason to expect that teaching will do good at a level better than chance. And this may not be far from the way things are.

So now let me offer part of my own model of schema construction.

SCHEMA CONSTRUCTION

SCHEMA BUILDING	SCHEMA TESTING
Mode 1 from our own encounters with actuality: experience.	Mode 1 against expectations of events in actuality: experiment.
Mode 2 from the schemas of others: communication.	Mode 2 comparison with the schemas of others: discussion.
Mode 3 from within, by formation of higher order concepts: by extrapolation, imagination, intuition: creativity.	Mode 3 comparison with one's own existing knowledge and beliefs: internal consistency.

Figure 1. (from Skemp, 1979)

As I use the term, construction includes both building and testing. The left hand column describes ways by which a schema is changed; the right hand column, ways in which we check whether a change is for the better. These are the methods which our intelligence has at its disposal for schema construction: and they work best when used in combination. The question is now, to what extent do they describe the methods by which mathematics is learnt in school? As I continue, you may like to fill in your own assessments in the table below, allotting marks from zero to ten.

	USED AT ALL		USED WELL	
	Building	Testing	Building	Testing
Mode 1				
Mode 2				
Mode 3				

Table 1

Building mode 1. This is used freely in the first few years of school, and then usually replaced by work on paper, which is seen as its purpose. For example, structural material such as multibase, abacus, are used for developing the concepts involved in place-value notation, and as a means for obtaining answers to questions such as 375 + 283, 375 - 283. The aim is for children to be able to do without help of this kind. Mode 1 building is often put to good use in these early years, and my complaint is only that it is discarded too soon.

Testing mode 1 is the reverse process - using a mathematical model to predict an outcome with physical materials and events, and then putting this prediction to the test. This may be done as a means of checking up on one's model, as for example by dropping a stone and a feather simultaneously in vacuum. This is the world of the experimental scientist, heavily dependent on mathematical models. Or it may be as a means for achieving a goal in the physical world. For example, if we fly on such and such a bearing we expect to arrive at Newfoundland. Again, the navigator is heavily dependent on a mathematical model.

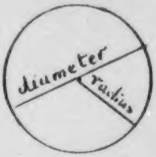
Certainly, experimental error can be a source of difficulty if we want to introduce mode 1 testing into the classroom. But my complaint is that teachers do not even try. They don't use mode 1 testing even when experimental error would present little or no problem. If structural materials are used to build mathematical models, it is often possible to use these models to make and test predictions of outcomes with these physical materials. In some initial trials of this I have been impressed by the pleasure shown by children when their prediction was confirmed by the outcome.

Overall, I would see the development of mode 1 testing as closely related to the development of closer links between mathematics teaching and science teaching.

Mode 2 building. This is the main method used in the majority of classrooms. Usually it is done very badly. Here is an example from a text book which is popular in England. It comes in a chapter on area. This began by explaining why, for a rectangle,

area = length x breadth,

followed by exercises in calculating areas of figures which were either rectangles, or could be divided into rectangles. Without any further explanation:

> Circles
>
> The circumference of a circle (that is its perimeter, or the length of its boundary) is found by measurement to be a little more than three times the length of its diameter. In any circle the circumference is approximately 3.1416 times the diameter, which is roughly $3\frac{1}{7}$ times the diameter. Neither of these figures is exact, as the exact number
>
> 
>
>
> cannot be expressed either as a fraction or a decimal. The number is represented by the Greek letter π (pi).
>
> Circumference = πd or $2\pi r$
>
> Area = πr^2

Examples of this kind of "telling, not teaching" abound. Contributions for my collection will be welcome!

What is wrong with the approach which this typifies, and what needs to be done instead for communication to be a successful contributor to schema building, I have discussed at length elsewhere (Skemp, 1971).

Testing mode 2. This refers to the testing of schemas. Indicating, e.g. by a tick or a cross, whether an answer is right or wrong is not testing mode 2. Nor is indicating whether a method is correct necessarily testing by mode 2, since it may be confined to showing whether or not a rule has been correctly followed. If however a method is discussed in relation to a parent schema, this I would call mode 2 testing, in combination with mode 3 - consistency with one's own existing knowledge and beliefs. If we consider this a little further, we shall see that fruitful discussion depends on the presence of shared schemas, or the discussants are not talking about the same things.

A very basic way of helping to ensure that discussants are talking about the same thing is to bring in mode 1 testing also. Thus, both discussants will be looking at the same physical objects and events. Even so, a shared schema is still necessary also, for without this they will not give the same interpretations to what they see.

There are two other features of discussion which make it valuable in schema construction. Successful communication involves the attachment of symbols and the clear formulation of one's ideas as preliminaries for communication, and these in turn favour reflection. The need sometimes to defend one's ideas also promotes reflection, with the object of being in a position to justify one's ideas in relation to an accepted body of knowledge.

Thus I would see discussion as a most valuable mode of schema construction, especially when combined with mode 3. And since mode 3 is the most sophisticated kind of testing, I would wish sometimes to support discussion with mode 1 testing.

To what extent do we find discussion of the kind I have been describing - a genuine interaction of mathematical ideas, of the kind we ourselves enjoy with colleagues and graduate students - in school mathematics lessons? . Before you answer, bear in mind that this would involve teachers listening to children's ideas (not just checking their answers), and children listening and responding to each others' ideas. How often do we find this taking place in the average mathematics lesson?

Building mode 3. The two clearest examples of this in mathematics learning are extrapolation, and the formation of higher order concepts. Much of mathematical learning requires one or both of these activities. Extrapolating is done by teachers all the time, but seldom in such a way that the pupils also extrapolate in their own minds, since they are not told what is happening. Consider multiplication. First they learn to multiply the natural numbers. (Often this is taught as repeated addition, which makes the concept particularly hard to extrapolate.) Then they are told how to multiply positive and negative integers, fractions, matrices. What they are not told is that "multiplication" now means something different. The concept itself has been extrapolated, and not just the method for doing it. Without this essential information it is unlikely that pupils will achieve relational understanding of this new meaning. So when they are asked "Which is larger, 2n or n+1?" they give replies such as "2n, because multiplying always makes bigger." (Hart, 1981).

The formation of higher order concepts involves the process of repeated abstraction: the discovery of regularities among objects and events which are not physical, but mental, objects and events. The constructivist principle shows particularly clearly here, since every learner has to make these discoveries afresh. Yet the need for good teaching is also here particularly great, since even the most gifted pupils, leave alone the average, are unlikely to be able to repeat, unaided, and in a few years, discoveries which have taken some of the best brains of mankind collectively many centuries to discover. So we have a paradox. Children can't be told this knowledge, and they can't construct it unaided. To resolve this paradox, children need teachers who are able indirectly to direct the mental creativity of others. How many teachers of this kind do you know?

If your answer is the same as mine, more's the pity, because mode 3 building is one of the most exciting and powerful activities in mathematics when it happens successfully. By helping it to happen, teachers might gain from their pupils real gratitude, and even devotion.

Testing Mode 3. This is quite a sophisticated activity, involving reflection not only on mathematical ideas in themselves but on relationships of implication between them. (If ABC is a triangle in which $A\hat{B}C = A\hat{C}B$, then ...). Formal proofs are an example of this in its fully developed form. Memorising formal proofs (yes, this does happen) is an example of its gross mis-use. Its proper use would be greatly encouraged by discussion, initially quite informal, in which the participants together examined statements, methods, results, using consistency with accepted mathematical ideas as criterion for acceptance or refutation. For this, relational understanding is a necessary but not sufficient condition. By taking part in this activity over a period of years, pupils might arrive at a stage where the activity itself, independent of its particular content, could be conceptualised and reflected on. Only in this way are they likely to reach an understanding of the nature and purpose of formal proof. I doubt if this is achieved very often. As a good use of mode 3 testing, I would certainly also accept the less sophisticated activity of informal discussion, which I see as a

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valuable means of leading pupils to an intuitive understanding of the nature of proof. How much of this takes place? I suspect that it varies greatly among classrooms. In those where text books and work cards are the predominant method of teaching, there can be very little.

* * * * *

Now to the uses of mathematics, as experienced by children in schools. I said earlier that not only were the uses different from those in the adult world, but the kinds of use. The importance I attach to this is based on the assumption that we construct different kinds of schema for different kinds of use. Hence, only the right kinds of use will favour the construction of sound mathematical schemas. Again, you may like to record your own assessments as we go along.

Uses of mathematics as experienced by children in schools.

(i) to achieve goals in the physical world
(ii) to co-operate with others
(iii) to create.
Or do they experience it as
(a) a job to be done
(b) a threat to be averted
(c) an exercise in fulfilling teachers' expectations
(d) a set of rules for making marks on paper?

Table 2

In group (i), mathematics provides an inexhaustible source of models from which particular plans of action can be derived for the achievement of a wide variety of goals in a wide variety of conditions. Even the same model can, by changing the units, be put to a variety of different uses. E.g.

$$d = vt \qquad D = kd \qquad E = IR$$

are these different uses of the same mathematical model.

In your judgement, is this how mathematics is made to appear to school children? Does it actually give them greater power to predict and control their environment, in their own here-and-now experience? Please consider your verdict.

Uses in group (ii) relate to the interpersonal, the social uses of mathematics for co-operation in science, technology, industry. Again, I can find no analogous use in the classroom. The kinds of co-operation which take place (e.g.) between designer and engineer, air navigator and ground controller, lens designer and computer programmer, just are not found.

Uses in group (iii). Here I refer to the uses of mathematical schemas as agents of their own growth. By reflecting on our schemas, extrapolating, imagining "What if ...", and sometimes also by intuitive leaps, existing knowledge can be used to create knowledge. It is closely related to mode 3 building. Though particular characteristic of pure mathematics, there can be a payoff also in mode 1. I find it exciting that an extrapolation of the idea of square root to that of a negative number has useful practical applications in electronics. Can we find similar excitements for children?

* * * * *

Having filled in my own assessments in tables 1 and 2, I find the results both depressing and hopeful. Depressing, because it seems to me that we're giving children a learning task which makes considerable demands on their intelligence, but we are not providing them with a situation in which they can use their intelligence to best effect. Rather the opposite. We are asking them to make an important and difficult intellectual journey, not just on level ground but a climb to considerable heights of abstraction, with their legs and their hands tied. I am hopeful also: for if we untie them, there is reason to think that we shall be pleasantly surprised at the progress which children then make. But the changes which will be necessary are more than changes in curriculum: they are major changes in our conception of what is a good learning environment. And if the majority of teachers are themselves the results of learning situations of the kind I have described, how can we persuade them first to untie themselves?

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COMPLEMENT AU VOLUME I DES ACTES

PSYCHOLOGICAL BLOCKS TO LEARNING OF MATHEMATICS IN REENTRY WOMEN

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ABSTRACT

Les campus universitaires des Etats-Unis voient aujourd'hui à l'intérieur de leur domaine un nouveau groupe d'étudiants: des femmes qui choisissent de recommencer leur instruction universitaire quand elles sont déjà d'un certain âge. Un grand pourcentage de ces femmes souffrent des angoisses à propos des cours de "maths" requis, et elles perçoivent les unités de valeur en mathematiques comme un obstacle assez décourageant quad il s'agit de réaliser leurs projects. Des tests psychologiques administrés à un groupe de ces femmes, qui ont participé dans des travaux pratiques destinés à enlever leurs angoisses à propos des "maths,", ont, en réalité, montré qu'il s'agissait de' hauts niveaux d'inquiétude quand elles devaient faire face à des phrases contenant des termes de mathématiques comme symbolisme. Les résultats ont aussi indiqué la préférence de ces femmes pour des renseignements absorbés mieux par la vue que par le son, ainsi qu'une préférence pour la participation plus que pour l'observation. Les processus de raisonnement de ces femmes avaient tendance à refléter l'application des règles apprises auparavant. Le laboratoire contre les angoisses au sujet des mathématiques était conduit par des psychologues et par des enseignants de mathématiques qui travaillaient ensemble. Le contenu du laboratoire se concentrait sur la capacité des femmes de survivre sous pression et sur leur façon de résoudre leurs problèmes dans une ambiance peu menaçante. Les évaluations faites après indiquaient un effort positif de la part de ces femmes pour dominer leurs angoisses.

College campuses in the United States today are seeing within their student body a new group of students: women who choose to resume their education in their middle years ("reentry" women). These women, many of whom have raised a family, are now returning to school with great determination to pursue a new and meaningful career. San Diego State University, a large public university in Southern California recently conducted a new student orientation program especially for reentry students. A large percentage of the women indicated at that time that they would like to pursue a career that involves mathematics, but they perceived this need for mathematics as a formidable roadblock in their plans.

To assist these women in achieving their goals, several women faculty organized a workshop modeled after the experiences reported by Shelia Tobias. Nineteen reentry women participated in the week-long workshop which met for two hours each day. The women ranged in age from 27 to 62 with a median age of 40.5 and an average age of 44. The women entered San Diego State University in the Fall of 1980 after a sustained absence from academic life. Their declared majors are in the areas of Social Work, Speech Pathology, History, Psychology, Anthropology, Zoology, Microbiology, Nursing, Art, Business Administration, and Liberal Studies. Eleven of the group are currently registered in a mathematics course.

In conjunction with the workshop each woman was administered a mathematics anxiety rating scale as well as a cognitive style mapping questionnaire. Results of the math anxiety scale indicated that a high percentage of these women were comfortable with doing arithmetic operations or problems where an arithmetic operation is clearly indicated. However, many of them showed high levels of anxiety on statements where a reference was made to mathematics using symbolism. Nearly all of them registered high levels of anxiety when a reference was made to evaluating their mathematical skills.

The cognitive mapping questionnaire indicated that certain personality and learning characteristics could be identified as being typical of this group of reentry women. Identification of their characteristics served not only to describe the group and support initial perceptions, but also to point out possible reasons for learning difficulties and/or anxieties as well as to indicate activities which could be incorporated into the workshop and from which the women would be likely to profit.

This group of reentry women endorsed statements reflecting a strong commitment to personal principles, a good knowledge of themselves, an awareness of time, reflected in a sense of the importance of meeting time deadlines and an ability and/or a preference for working independently on projects or problems. All of these attributes are consistent with characterizing this group as being dedicated, serious students with clear goals to which they are committed.

These women also tended to be uniform with respect to certain learning characteristics which are related to academic pursuits. They tended to prefer visual input for both verbal and numeric information. Furthermore, input of numeric information by hearing was overwhelmingly the least preferred modality of information processing. Of the five senses, information processing by the sense of touch was predominately preferred to any other of the five senses. They also indicated a strong preference for learning by doing rather than learning by observing. They tended to take a non-assertive role in their relationship with others, avoiding or feeling uncomfortable with a role which called for persuasion or convincing others of their viewpoint. They tended to deemphasize the role of their peer group in influencing their actions or providing assistance for them. Finally the reasoning processes of these women tended to reflect primarily the application of previously learned rules.

The workshop organized to assist these women was conducted by three professionals trained in psychology or counseling and two mathematics educators. Activities were divided approximately evenly between the counseling group and the mathematics group.

The counseling group presented activities which were designed to provide the women with experiences in managing stressful situations and to encourage acknowledgement of their anxieties. These included activities such as a demonstration of relaxation techniques, a discussion of the socialization of women and its possible effects on their attitude toward mathematics, clues on overcoming test anxiety and a discussion of cognitive restructuring of self-imposed belief systems.

The mathematics group engaged the women in activities designed to convince them of their ability to learn unfamiliar mathematical language as well as some problem-solving techniques. Geometric transformations were used in a very intuitive manner to create figures which would tesselate a plane and several geometric theorems were demonstrated by using transformations. Games and mathematic puzzles were used to demonstrate some of the problem- solving techniques found in Polya's work. Activities were chosen which provided the women with an opportunity to seek solutions by manipulating objects or drawing diagrams. Interactive participation by the women was encouraged and did, indeed, take place.

During the final sessions of the workshop each woman conferred with a member of the staff who provided an interpretation of the cognitive style mapping administered earlier in the week. Each also had a conference to assess their mathematics needs. Following the workshop tutors were provided for those women currently enrolled in a mathematics course. Care was taken to insure that the tutors were cognizant of the special problems of reentry women. This group of women have subsequently formed the nucleus of a network designed to provide support for reentry women.

An evaluation instrument administered at the conclusion of the workshop indicated a very high degree of satisfaction on the part of the women. The statement in the evaluation which elicited the greatest positive response was "The workshop gave me a basic feeling of support." A follow-up investigation of these women currently registered in a mathematics course will be done at the conclusion of the Spring term to determine their accomplishments and possible changes in attitude toward mathematics.

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COGNITIVE FUNCTIONING, CLASSROOM LEARNING AND EVALUATION: TWO PROJECTS

by

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ABSTRACT

Dans ce mémoire, les auteurs veulent discuter les progrès de deux études collaboratives.

Le premier programme s'intéresse au rapport entre la faculté de transformations cognitives (CPC) et le niveau de réussites aux problèmes d'addition et de soustraction pour un groupe d'enfants agés de 4 à 8 ans en Tasmanie en Australie. D'abord, on a groupé les enfants selon leur CPC--un indice composé des résultats de deux séries de tests, le premier qui a mesuré "M-space" et le deuxième qui a mesuré le nivean du développement cognitif. Ensuite, on a examine le rapport entre le groupe dans lequel l'éstudiant a été placé et sa réussite à propos d'autres variables. Les résultads préliminaries mon trent que:

(i) il y a une augmentation significative du niveau de réussites sur les problèmes mathématiques qui accompagne chaque augmentation du CPC.

(ii) les enfants de CPC différent utilisent des stratégies differentes.

(iii) le niveau de travail actif ("engagement") s'clèvè selon l'augmentation du CPC.

(iv) les étudiants ayant un CPC élevé s'appliquent aux tâches quelle que soit l'activité du professeur.

(v) le pourcentage d'étudiants qui utilisent les algorithmes enseignés pour trouver la solution d'un problème ne s'élevè as selon l'augmentation du CPC.

Le deuxième programme de recherches vise au développement d'une série de "super-items" (Cureton, 1965) pour examiner la capacité des étudiants agés de 8 à 17 ans à résoudre des problèmes mathématiques. La technique utilisée est fondée sur un procédè qui vient d'être developper (Biggs & Collis, 1981); ainsi, on peut analyser les réponses des étudiants selon les caractéristiques de leur structure. Dans ce programme, on a renversé le procédé habituel: on cherche à développer des questions d'une structure particulière afin que la réponse correcte soit un indice d'une aptitude à résoudre les problèmes mathématiques d'un tel niveau.

Nous avons choisi les critères suivants selon les quels nous avons développé les quatre questions posées vis-à-vis "la tige" de chaque problème.

1. une, seulè structure: l'étudiant doit utiliser un seul renseignement qui est presénté d'une facon bien claire dans la tige.
2. plusieurs structures: l'étudiant doit utiliser au moins deux opérations complètes mais differentes à propos des renseignements séparés dans la tige.
3. un rapport: l'étudiant doit utiliser au moins deux opérations complètes mais differentes qui montrent une compréhension intégrée de la plupart des renseignements dans la tige.
4. une abstraction extensive: l'étudiant doit utiliser un principe abstrait qui est évoqué par les renseignements dans la tige entiere.

Au moment actuel, on vient de développer les problèmes et en automne on espère les examinee d'une facon empirique a propos d'un èchantillon assez large d'étudiants américains.

* *
*

To this point, my attention has been devoted mainly to children's processing capability whereas my co-author, Professor T. A. Romberg, has had extensive experience in experimental classroom research and analysis. The joint projects that we have undertaken have underlying them the notions that not only might more rapid improvement in classroom instruction result from the collaborative approach but also that new light might be shed on fundamental issues in our own special areas of interest. Currently, we are engaged in two joint projects which I wish to report in this paper. The first, and larger of the two projects, is still being analyzed but enough has been done to give a preliminary flavour of what the final results are likely to be. The second has only just reached the end of stage 1 of a two-stage project but the approach and preliminary results are of sufficient interest to warrant reporting at this conference.

Project No. 1: Cognitive Level and Performance on Addition and Subtraction Problems

The intention in this project was to endeavour to classify a population of children aged 4 to 8 years into groups according to their Cognitive Processing Capabilities (CPC)--this was done by giving two batteries of tests one to measure M-space (Case, 1978) and one to measure Cognitive Developmental Level. The tests in both batteries were intended to bear implicitly on the early learning of mathematical material. The CPC measures and their descriptions were to be obtained by combining the information obtained from the two batteries of tests. Following this classificatory procedure, the relationship between each distinct group with particular CPC characteristics and various experiences incorporating mathematical content in the beginning school years was to be examined. A preliminary examination of some of this data was reported by my co-author at Berkeley last year (Romberg & Collis, 1980a). Let us summarize the results of grouping according to CPC measures.

It was found that, using Factor Analytic techniques and a cluster analysis procedure (for details, see Romberg & Collis, 1980b, c), the population of 4 to 8 year olds could be assigned to six groups with the associated characteristics summarized in Table 1.

The tables following Table 1 indicate the direction in which the analyses are leading us with respect to the relationship between CPC level and the following five variables:

(i) achievement on elementary addition and subtraction problems,
(ii) strategies used by pupils on elementary addition and subtraction problems;
(iii) pupil engagement on school tasks;
(iv) teacher actions: the effect on pupil engagement;
(v) pupil use of addition and subtraction algorithms.

All the results quoted below are from the total population and are independent of both age and grade or class.

Table 1

M-space Groupings with Associated Characteristics

Group	M-space Measure	Characteristic
1	1	elementary qualitative comparisons only, lack quantitative and logical ability
2	2	qualitative correspondence, lack specific quantitative and logical skills
3	2 *S+	high qualitative correspondence, have certain specific quantitative skills (i.e., counting for specific purposes), do not reach criterion on logical skills
4	3 *S-	high qualitative correspondence, high on quantitative skills, do not reach criterion on logical skills
5	3 *S+	ceiling on qualitative correspondence, high on quantitative skills, high on logical skills
6	4 *S-	

*S+ and S- represent the presence/absence respectively of a spatial ability as measured by one of the tests.

(i) Achievement: Table 2 shows the percentage correct by CPC level on addition and subtraction problems using numbers up to 20. The results are the combined scores for two tests, one in which physical material was available and one in which physical material was not available. The results show a significant increase in achievement by CPC level--the biggest gains being made between levels 1 and 2 and again between levels 2 and 3.

(ii) Strategies: Table 3 shows the percentage of the various kinds of strategies used by the same children on the same problems as were involved in the results in Table 2.

Table 2

Achievement on Tests C± Tasmanian Data
(% Correct, Total Population)

CPC Level	% Correst Responses	No. of Trials Involved
1	22	180
2	65	450
3	81	396
4	83	264
5,6	96	252
Total	72	1542

Table 3

Pupil Strategies on Tests C± Tasmanian Data
(% of Times Strategy Used; Total Population)

CPC Level	Direct Modeling[1]	Counting	Routine Mental Operation[2]	Non-routine Mental Op.[3]	Inappropriate
1	28	0	1	0	70
2	36	18	13	6	27
3	18	33	26	10	13
4	11	30	35	14	9
5,6	13	40	42	6	0

[1] Using physical material, e.g.. counters, fingers, etc.

[2] Using known number facts or relationships

[3] Innovative use of number facts or relationships

There are several interesting features of Table 3 which need careful consideration, three of which will be mentioned here. First, there is a very significant drop in the use of inappropriate strategies employed from levels 1 through to level 3; second, the use of direct modeling goes down as CPC level rises and the use of counting and routine mental operations rises; third, the reduction in use of inappropriate strategies is spread over the other categories, with counting and routine mental operations taking the largest and almost [illegible] shares.

[illegible] Pupil Engagement: Table 4 gives us information on the percentage of class time during which the children at the various CPC levels are engaged on the school task in hand.

Table 4

Pupil Engagement on Task
Tasmanian Classroom Observational Data
(% of Time Pupil on Task; Total Population)

CPC Level	% Time On Task	% Time Off Task
1	64	36
2	65	35
3	70	30
4	76	24
5,6	87	13

It appears from these results that those pupils who can best afford to be off task spend more time in this state than their colleagues with higher CPC levels. The data is being broken down for more detailed examination at this time and will be reported more fully in our final report. However, one piece of analysis which is available and is of interest in this context is that when engaged the level 1 group spend 86% of this time on content and 14% on being given directions as to what to do--the percentages in the same categories for level 5, 6 are 91% and 9%. In other words, there does not seem to be a lot of difference between the levels in what the engaged time is spent on but there is a large difference in the amount of time spent attending to classroom tasks from level to level.

(iv) Teacher Action and Pupil Engagement: Table 5 gives us information on percentage of time that the pupil remains engaged on his/her classroom task when the teacher does various things.

Table 5

Teacher Action vs. Pupil Engagement on Task
Tasmanian Classroom Observational Data
(% of Time Pupil on Task; Total Population)

CPC Level	TEACHER ACTION				
	Speaking to Individual Pupil	Speaking to Small Group	Speaking To Large Group	Not Speaking But Interacting	Not Speaking Not Interactin
1	67	55	65	62	51
2	61	71	77	63	60
3	69	82	78	63	66
4	73	72	87	76	76
5,6	79	92	95	89	91

It can be seen from this summary that regardless of what the teacher is doing the children with higher CPC levels tend to spend more of their time on the school task in hand. In addition, some of the patterns of behaviour suggested in the table may repay closer examination, e.g., the teacher not speaking nor interacting with anyone seems to have the greatest distraction from task engagement for the lower CPC level pupils.

(v) Pupil Use of Algorithms: Table 6 shows the percentage of children, by CPC level, who, having learned the algorithms for addition and subtraction, actually used them to obtain the answer to a problem.

Table 6

Use of Known Algorithm to Solve Problem Tasks D and E
Tasmanian Data
(% Using Algorithm, Total Population of Subjects Who Had Learned Algorithm)

CPC Level	% Using Algorithm	% Using Inappropriate Strategy	% Using Counting
2	24	37	8
3	19	19	22
4	20	19	18
5,6	25	3	32

Children at all CPC levels use the taught algorithm infrequently, between one-fifth and one-fourth of the number of times when it is appropriate. They appear to prefer to fall back on more "primitive" strategies such as counting which they have used successfully previously.

It can be seen that the data on these tests parallel those in Table 3 in that, with rise in CPC level, the use of inappropriate strategies decreases significantly at the same time as use of counting strategies increases. It is of interest to note that when the children cease to use inappropriate strategies they do not, in the main, turn to the algorithm which has been taught as the appropriate strategy. In fact, for this population, the use of the algorithm does not increase significantly with increasing CPC level. It is interesting to speculate on the reasons for this. Perhaps the emphasis on understanding the relationship between the algorithm and its application is misplaced at least at this early stage; perhaps we should treat problem solving strategies and algorithmic procedures as discrete entities, teach them separately and worry about bringing them together at a later stage in the child's mathematical development.

Project No. 2: Cognitive Level and Assessment of Students' Problem Solving Mathematical Capability

This project has set out to combine an evaluation technique designed by two Australian researchers (Biggs & Collis, 1981) with a mathematical item type proposed sometime ago by Cureton (1965) in order to devise multiple-choice items which could be used to test validly a student's capacity for solving mathematical problems. Based initially on the cognitive development literature, Collis and Biggs devised a response model which enabled a child's response to a particular question to be classified according to the way in which it was structured. The response model or SOLO* Taxonomy proposes that there are five basic categories within the concrete operational mode of functioning (i.e., 7 + years to 15 + years). These basic categories* are set out diagrammatically in Figure 1.

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*SOLO: Structure of Learned Outcome

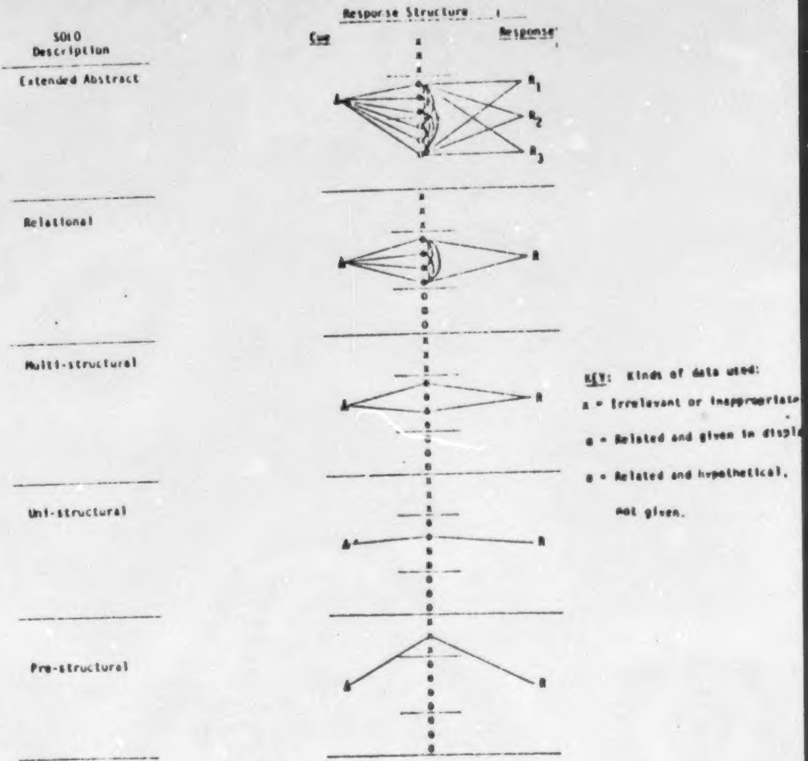
SOLO Description
Extended Abstract
Relational
Multi-structural
Uni-structural
Pre-structural
Response Structure
Cue
Response
A
R1
R2
R3
R
KEY: Kinds of data used:
x = Irrelevant or inappropriate
● = Related and given in displ
○ = Related and hypothetical, not given.

Figure 1. SOLO response description.

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*It should be noted that Biggs and Collis (1981) do not confine their use of this model to mathematics--it can be applied across the whole curriculum. Moreover, the model is equally applicable in other modes of functioning, viz., sensori-motor, pre-operational and formal--the concrete mode is dealt with here because it covers the range of compulsory schooling in most countries. For full details, the reader is referred to Biggs, J. B., & Collis, K. F. Evaluating the Quality of Learning: The SOLO Taxonomy, Academic Press Inc., New York. (In press, expected release Nov. 1981.)

The above diagram is meant to cover the general case. For particular content areas, certain idiosyncracies peculiar to the area need to be taken into account. For mathematics, the following can serve as a highly condensed summary of a response model which is meaningful within the context of school-based mathematical material.

Summary of Response Modes

UNI-STRUCTURAL RESPONSES

Marked by a *single direct relationship* to concretely (either physically or iconically) available criteria.

MULTI-STRUCTURAL RESPONSES

The ability to handle *multiple operations* with small numbers by a *series of meaningful closures*, for instance, may be seen as analogous to using a sequence of given, but unconnected, propositions to support a particular judgment in other content areas.

RELATIONAL RESPONSES

The individual *relates elements within the* immediately available concrete *system* and forms generalizations on this basis.

EXTENDED ABSTRACT RESPONSES

Acceptance of lack of closure, use of the reciprocal operation and ability to work with multiple interacting and abstract systems involve a *comprehensive use of the given data together with related hypothetical constructs* and abstract principles.

Superitems

Cureton (1965) seems to have been responsible for coining this term to describe sets of questions which were asked about a particular problem situation. Typically, the problem situation would be described in the stem which would consist of a paragraph describing the problem and the items would consist of a series of questions which could be answered by reference to the information in the stem. Cureton's basic interest was methodological but others (e.g., Wearne & Romberg, 1976) have used the notion since to develop tests of mathematical problem solving. This latter work showed that the tests were useful because they provided more information about the student as well as a more refined measure of the child's problem-solving ability. However, they did not give information about the level of a child's reasoning in respect of each problem situation. It is this latter aspect that this project is designed to shed light on.

It is hypothesized that, by using the SOLO technique in reverse, as it were, one ought to be able to design items such that a series of questions on the stem would require a more and more sophisticated use of the information in the stem in order to obtain a correct result. This increase in sophistication should parallel the increasing complexity of structure noted in the SOLO categories.

Criteria for Construction of Superitems

Clearly the construction of these items consists of two parts, (a) the writing of the stem and (b) the construction and writing of questions to reflect the SOLO levels. The former is concerned with content validity. This has been achieved in the present study by outlining six items for each of the content areas set up by the National Assessment of Educational Progress (NAEP) publication on mathematics objectives (NAEP, 1977) and carrying out several content vs. category checks using math teachers, educators and mathematicians. The latter requires that suitable criteria be set up to enable four questions to be asked which would not only require a knowledge of the information in the stem but would also be such that a correct response to each question would be indicative of an ability to respond to the information in the stem at least at the level reflected in the SOLO structure of the particular question. To achieve this last the following criteria were set up for designing the questions:

1. Uni-structural: Use of one obvious piece of information coming directly from the stem.
2. Multi-structural: Use of two or more discrete closures directly related to separate pieces of information contained in the stem.
3. Relational: Use of two or more closures directly related to an integrated understanding of the information in the stem.
4. Extended Abstract: Use of an abstract general principle or hypothesis which is derived from or suggested by the information in the stem.

In each superitem, the correct achievement of question 1 would indicate an ability to respond to the problem concerned at at least the uni-structural level. Likewise success on question 2 corresponds to an ability to respond at multi-structural level and so on.

Conclusion

The above is meant to outline the first stage in the development of a critical aspect of a set of superitems to test mathematical problem solving. The next stage will be to subject the hypotheses generated to empirical test and this will be undertaken in the Fall semester of this year.

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MATHEMATICAL VERNACULAR : AN INTRODUCTION

Jan Donkers

1. Introduction

In studying problems of language in mathematics education one frequently comes across discussions about problems that are mainly of a socio-psychological character. As it is indicated in a survey paper by Austin and Howson [1], little research has been done about formal aspects of language in mathematics education. In this paper we shall deal with such formal aspects. These formal aspects apparently have a more than formal significance.

We shall briefly discuss the so-called Mathematical Vernacular, sometimes abbreviated as M.V. By this we mean a system of rules and conventions that enables one to present a mathematical text in an umambiguous manner and such that the structure becomes immediately clear.

The ideas have originally been developed by Prof. N.G. de Bruijn in his lectures on Language and Structure of Mathematics at the University of Technology in Eindhoven (cf. [2], [3]). In Eindhoven we have started a studygroup to elaborate De Bruijn's ideas and to gain some experience in applying his ideas to mathematical texts (cf. [4]).

This paper discusses some experiences with the translation of mathematical texts and describes a few of the main rules, notations and conventions.

We shall do this by means of an example: a translation into M.V. of a mathematical text (see Appendix 1). We have chosen a part of text from Rudin's Principles of Modern Analysis (see Appendix 2), because it demonstrates some features of M.V. quite well and because it appears to be a well known and clearly written book, which, in our opinion, is frequently used in the English speaking countries.

This text is called Rudin's text. It is not exactly his text because we have put it together by cutting and pasting from his book. We are mainly interested in the proof of theorem 1.20(a), the so called archimedian property for the reals.

In the proof some definitions and a part of a theorem are used, so there is a translation of these as well. Translated are only those parts of the definitions which are relevant to the proof. So we have examples of two important specimens of mathematical texts, namely a definition-text and a proof-text.

We do not intend to criticise the book nor the part of text that we have taken from it. We only use this as a typical example to illustrate the main features of M.V.

2. Some general remarks

We shall make a few remarks to begin with.

(1) We use the well known logical operators $\neg$, $\wedge$, $\vee$, $\Rightarrow$ and the logical symbols $\forall$, $\exists$, and $\exists!$ (the last one for unique existence).

(2) Everything in M.V. is written in some context. What we mean by context is the following:

(i) all the introduced variables, still valid at a given moment (every variable with a description of its meaning by specifying its domain)

(ii) all the assumptions, valid at a given moment.

To be able to read (and understand) a text, it is necessary that the words and symbols which have been defined previously are known. Such words and symbols are not part of the context. For example: the symbol $<$ stands for an order relation on a set, P(A) means the power set of a set A; $\in$ has the meaning "is element of", etc.

We mark the beginning as well as the end of the lifetime of every assumption and every variable as follows: a variable is introduced by using a pointed flag and the introduction of an assumption is made by using a rectangular flag. The length of the flagstaff gives the duration of the validity (briefly the lifetime) of the variable or the assumption respectively.

(3) Several times we used the word translation, but by this is not meant a literally translation. So it would be better to speak of rewriting the given text in M.V. At some places the M.V. text differs from Rudin's text, but we shall come back to this point later.

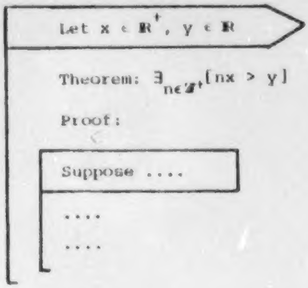
3. The context-structure

On the first two lines of the M.V. text we see two pointed flags. In the first flag a variable-pair of type "ordered set" is introduced and in the second flag a variable of type "subset of S".

On line 8 as well as on line 18 there is a rectangular flag, marking the introduction of an assumption. Note that in contrast with the pointed flags, in which it may happen that symbols appear for the first time, nothing new is inside a rectangular flag. This means, every symbol word or sentence, used inside a rectangular flag is already known, because it has been defined previously. Looking at the flag-structure, we see that the staff of the first flag spans lines 1 to 11. The other two flagstaffs of the first 11 lines end on line 10. They mark the context. So on every line it is immediately clear what is in the context (of that line). On lines 12 and 13 the context is empty.

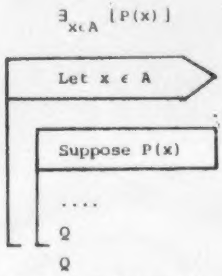
What happens if a flagstaff ends? A characteristic of a pointed flag is that it can be followed by a generalization. Indeed, the sentence following the end of a flagstaff of a pointed flagg often starts with "for every ..." or with a universal quantifier. For an example we refer to line 33.

From Rudin's text it is not clear whether we have to write theorem 1.20 (a) with or without a context. The proof starts using x and y as if they were mentioned before. So it suggests a non-void context (that means, implicity using a pointed flag). On the other hand, the theorem is stated by means of an implication and suggests a generalization. It is quite usual in mathematical texts, to state a theorem in a non-void context. In a non-void context, the M.V. text of theorem 1.20(a) could be:


Let $x \in \mathbb{R}^+$, $y \in \mathbb{R}$
Theorem: $\exists_{n \in \mathbb{Z}^+}[nx > y]$
Proof:
Suppose ....
....
....

The end of a flagstaff of a rectangular flag is often followed by an implication. In case of a contradiction it is followed by the negation of the assumption, as is the case on line 32.

On line 26 appears a peculiar flag. It is a pointed-rectangular flag and is called a double (or mixed) flag. A double flag indicates what is called existential elimination. The general scheme for existential elimination looks like this:



On the line following the existential proposition, there appears a pointed flag with the introduction of the variable x with its type description. On the next line, the assumption P(x) is introduced inside a rectangular flag. Within these two context flags the proposition Q is derived, which does not contain the variable x. Hence we obtained, outside the flags, the following two propositions:

(1) $\exists_{x \in A}[P(x)]$

(2) $\forall_{x \in A}[P(x) \rightarrow Q]$

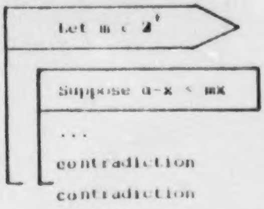
From these two propositions we may conclude to Q outside the context.

The scheme for the existential elimination is an important one, because it is applied frequently in a mathematical text. Many times a text is obscure, because one does not realize that one is dealing with existential elimination. Here is an example:

$$\exists_{x \in \mathbb{R}} [x > 1], \text{ then } x > 0 \ \ldots$$

In the M.V. text, there is an example of existential elimination on the lines 25 and following. This has to be read as follows:

25 then $\exists_{m \in \mathbb{Z}^+} [a - x < mx]$



Let $m \in \mathbb{Z}^+$

Suppose $a - x < mx$

...

30 contradiction

31 contradiction

Since existential elimination occurs so often and since $P(x)$ can be a long expression, it is easy to have a shorthand notation. We use the double flag with the standard text "Choose ... accordingly" for short.

4. Various kinds of definitions

The symbol ":=" on line 3 indicates a definition. In M.V. the things we define can be: statements, substantives and names.

The definition on line 3 is an example of a substantive definition. This is expressed by using the capital S, the so-called substantive binder (or quantifier). What is a substantive? In the sentence: "p is a ...", we use at the open place a substantive. For example: "p is a real number". We consider "real number" as a substantive instead of as being a compound of an adjective and a substantive. In "p is a positive real number less than 100", the combination "positive ... 100" is a substantive. Using the binder S, we write this substantive as

$$S_{x \in \mathbb{R}} \ (0 < x < 100).$$

The usage of the substantive binder helps to make a text clear. Sometimes it makes things easier to handle, as is the case with

$$\text{divisor of } n := S_{k \in \mathbb{Z}} \ (\exists_{m \in \mathbb{Z}} [km = n]) \ .$$

The usage of this binder S has the advantage of giving a clear signal.

As to the definition on line 3, one is probably more familiar with an expression like: "an element β of S is called an upper bound of E, if ... ".

The definition on line 4 is the definition of a name. The arrow is an operator that assigns a set to a substantive. So B is the name of the set of all upper bounds of E. One may write $B := \{x \in S | x \text{ is an upperbound of } E\}$ as well.
On line 5 appears the definition of the statement: "E is bounded above".
The name-definition on line 10 appears within more context than the other definitions. This expresses that one has to show that there is exactly one least upper bound of E. In that case we can speak of the least upper bound. So if the set of least upper bounds of E is not empty, then it is a singleton and we give its element a name: sup E.
In M.V. we don't accept things like "sup E exists" as in Rudin's definition 1.10, because we do not want to give names to objects which have not been proved (or explicitly assumed) to exist. As a consequence the M.V. text of line 11 deviates from Rudin's text.
It is obvious that the definitions made in the first part of the M.V. text may be used outside the given context. Sometimes an extra reference is needed, as is the case with the lines 20, 21 and 22.

5. Concluding remarks

Up to now, we have given a very brief explanation of the main notations and conventions in M.V.
In our opinion, using M.V. has certain advantages.

(1) It clarifies the context structure as well as the logical structure of a mathematical text, expecially with respect to definitions and proofs.
(2) It clarifies the way the language is used in a mathematical text (e.g. substantives - statements - names).
(3) It seems a relevant aid for the teacher when he is preparing a lesson. It can clarify some obscure passages in a mathematical text and raise some hidden thoughts to a level of consciousness.

It will be obvious that a textbook entirely written in M.V. would be a very dull one. M.V. is not well suited for communication between mathematicians when they discuss a mathematical problem, for example at the blackboard or by mail. Of course, during moments of confusion or misunderstanding in such discussions, one might want to use M.V. anyway. But we think, if one has some experience with M.V., one gets another view on mathematical texts. In any case, it is because of our experience with M.V., that in reading an ordinary mathematical text, we begin to see the difficulties earlier and more clearly.

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Eindhoven, May 1981

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1.   Let $(S,<)$: ordered set
2.       Let $E \in P(S)$
3.       upper bound of E := $S_{\beta \in S}\,[\forall_{x \in E}(x \leq \beta)]$
4.       $B$ := (upper bound of E)!
5.       E is bounded above := $B \neq \emptyset$
6.       least upper bound of E := $S_{\alpha \in B}\,[\forall_{\gamma \in S}(\gamma < \alpha \to \gamma \notin B)]$
7.       (clearly there is at most one least upper bound of E)
8.           Suppose E has a least upper bound
9.           then $\exists!_{\alpha \in S}$ [$\alpha$ is a least upper bound of E]
10.          sup E := the least upper bound of E
11.      $(S,<)$ has the l.u.b. property := $\forall_{E \in P(S), E \neq \emptyset}[B(E) \neq \emptyset \to$ E has a least upper bo
12. 1.19  Theorem: $(\mathbb{R},<)$ has the l.u.b. property.
13. 1.20(a) Theorem: $\forall_{x \in \mathbb{R}^+}\forall_{y \in \mathbb{R}}\exists_{n \in \mathbb{Z}^+}[nx > y]$
14.       Proof:
15.  Let $x \in \mathbb{R}^+$, $y \in \mathbb{R}$
16.  $A := \{z \in \mathbb{R} \mid \exists_{n \in \mathbb{Z}^+}[z = nx]\}$
17.  then $x \in A$, so $A \neq \emptyset$.
18.      Suppose $\neg\,\exists_{n \in \mathbb{Z}^+}[nx > y]$
19.      or $\forall_{n \in \mathbb{Z}^+}[nx \leq y]$
20.      then y is an upper bound of A     }
21.      hence A is bounded above          }  (with respect to $(\mathbb{R},<)$)
22.      define (th. 1.19) $\alpha$ := sup A  }
23.      then $\alpha - x < \alpha$
24.      hence $\alpha - x$ is not an upper bound of A
25.      then $\exists_{m \in \mathbb{Z}^+}[\alpha - x < mx]$
26.          choose m accordingly
27.          then $\alpha < (m+1)x$
28.          but $(m+1)x \in A$
29.          so $\alpha$ is not an upper bound of A
30.          contradiction
31.      contradiction
32.  $\exists_{n \in \mathbb{Z}^+}[nx > y]$
33.  conclusion: $\forall_{x \in \mathbb{R}^+}\forall_{y \in \mathbb{R}}\exists_{n \in \mathbb{Z}^+}[nx > y]$.
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1.7 Definition Suppose S is an ordered set, and $E \subset S$. If there exists a $\beta \in S$ such that $x \leq \beta$ for every $x \in E$, we say that E is *bounded above*, and call β an *upper bound* of E.

1.8 Definition Suppose S is an ordered set, $E \subset S$, and E is bounded above. Suppose there exists an $\alpha \in S$ with the following properties:

(i) α is an upper bound of E.
(ii) If $\gamma < \alpha$ then γ is not an upper bound of E.

Then α is called the *least upper bound of E* [that there is at most one such α is clear from (ii)] or the *supremum of E*, and we write

$$\alpha = \sup E.$$

1.10 Definition An ordered set S is said to have the *least-upper-bound property* if the following is true:

If $E \subset S$, E is not empty, and E is bounded above, then $\sup E$ exists in S.

1.19 Theorem *There exists an ordered field R which has the least-upper-bound property.*

1.20 Theorem

(*a*) *If $x \in R$, $y \in R$, and $x > 0$, then there is a positive integer n such that*

$$nx > y.$$

Proof

(*a*) Let A be the set of all nx, where n runs through the positive integers. If (*a*) were false, then y would be an upper bound of A. But then A has a *least* upper bound in R. Put $\alpha = \sup A$. Since $x > 0$, $\alpha - x < \alpha$, and $\alpha - x$ is not an upper bound of A. Hence $\alpha - x < mx$ for some positive integer m. But then $\alpha < (m + 1)x \in A$, which is impossible, since α is an upper bound of A.

From: W. Rudin, Principles of mathematical analysis (3rd edition).

QUERIES AROUND THE NUMBER CONCEPT

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Résumé :

Questions par rapport à la notion du nombre.

Il y a des problèmes irrésolus autour de la notion du nombre telle qu'elle est connue par les petits enfants.
Leurs idées se distinguent beaucoup des nôtres.
Comment découvrir les idées des enfants ?
Comment les comprendre ?
Comment appliquer ces résultats dans l'enseignement ?

On présentera trois sujets :

I. l'observation mutuelle qui est notre méthode de recherche ;

II. une énumération de sujets possibles de recherche qui nous frappaient ;

III. une discussion détaillée de trois parmi ces sujets :

- a) le comptage acoustique ;
- b) le comptage de mouvements au lieu de celui d'objets ;
- c) l'entendement enfantin du nombre
  décrit mathématiquement et psychologiquement.

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